Relations definitions: Let \( R \) be a relation on \( A \). In other words, \( R \subseteq A \times A \). Then:

- \( R \) is reflexive iff for all \( a \in A \), \( (a, a) \in R \).
- \( R \) is symmetric iff for all \( a, b \), if \( (a, b) \in R \), then \( (b, a) \in R \).
- \( R \) is antisymmetric iff for all \( a, b \), if \( (a, b) \in R \) and \( a \neq b \), then \( (b, a) \notin R \).
- \( R \) is transitive iff for all \( a, b \), if \( (a, b) \in R \) and \( (b, c) \in R \), then \( (a, c) \in R \).

Let \( R, S \) be relations on \( A \). Then:

- \( R \circ S = \{(a, c) : \exists b \text{ such that } (a, b) \in R \text{ and } (b, c) \in S\} \)

1. Relations Examples
   (a) Suppose that \( R, S \) are relations on the integers, where \( R = \{(1, 2), (4, 3), (5, 5)\} \) and \( S = \{(2, 5), (2, 7), (3, 3)\} \). What is \( R \circ S \)? What is \( S \circ R \)?

   (b) Consider the relation \( R \subseteq \mathbb{Z} \times \mathbb{Z} \) defined by \( (a, b) \in R \) iff \( a \leq b + 1 \). List 3 pairs of integers that are in \( R \), and 3 pairs of integers that are not.

   (c) Consider the relation \( R \subseteq \mathbb{Z} \times \mathbb{Z} \) defined by \( (a, b) \in R \) iff \( a \leq b + 1 \). Determine if \( R \) is reflexive, symmetric, antisymmetric, and/or transitive. If a relation has a property, explain why. If not, state a counterexample.
2. Relations Proofs
Suppose that $R, S \subseteq \mathbb{Z} \times \mathbb{Z}$ are relations.

(a) Prove or disprove: If $R$ and $S$ are transitive, $R \cup S$ is transitive.

(b) Prove or disprove: If $R$ and $S$ are reflexive, then $R \circ S$ is reflexive.

(c) Prove or disprove: If $R \circ S$ is reflexive, then $R$ and $S$ are reflexive.

(d) Prove or disprove: If $R$ is symmetric, $\overline{R}$ (the complement of $R$) is symmetric.
3. Constructing DFAs
For each of the following, construct a DFA for the specified language.
(a) Strings of $a$'s and $b$'s with odd length ($\Sigma = \{a, b\}$).

(b) Strings with an even number of $a$'s ($\Sigma = \{a, b\}$).

(c) Strings with an odd number of $b$'s ($\Sigma = \{a, b\}$).

(d) Strings with an even number of $a$'s or an odd number of $b$'s ($\Sigma = \{a, b\}$).
4. Structural Induction: Dictionaries

Recursive definition of a Dictionary (i.e. a Map):

- Basis Case: [] is the empty dictionary
- Recursive Case: If D is a dictionary, and \( a \) and \( b \) are elements of the universe, then \((a \rightarrow b) :: D\) is a dictionary that maps \( a \) to \( b \) (in addition to the content of \( D \)).

Recursive functions on Dictionaries:

\[
\text{AllKeys}([ ]) = [] \\
\text{AllKeys}((a \rightarrow b) :: D) = a :: \text{AllKeys}(D) \\
\text{len}([ ]) = 0 \\
\text{len}((a \rightarrow b) :: D) = 1 + \text{len}(D)
\]

Recursive functions on Sets:

\[
\text{len}([ ]) = 0 \\
\text{len}(a :: C) = 1 + \text{len}(C)
\]

Statement to prove:
Prove that \( \text{len}(D) = \text{len}(\text{AllKeys}(D)) \).
5. Structural Induction on Palindromes

Consider the following recursive definition of the set $B$ of palindrome binary strings:

- **Base case:** $\varepsilon \in B$, $0 \in B$, $1 \in B$.

- **Recursive steps:**
  - If $s \in B$, then $0s0 \in E$, $1s1 \in B$, and $ss \in B$.

Now define the functions $\text{numOnes}(x)$ and $\text{numZeros}(x)$ to be the number of 1s and 0s respectively in the string $x$.

Use structural induction to prove that for any string $s \in B$, $\text{numOnes}(s) \cdot \text{numZeros}(s)$ is even.