CSE 390Z: Mathematics of Computing

Week 9 Workshop

Conceptual Review

Relations definitions: Let R be a relation on A. In other words, $R \subseteq A \times A$. Then:

- R is reflexive iff for all $a \in A$, $(a, a) \in R$.
- R is symmetric iff for all a, b, if $(a, b) \in R$, then $(b, a) \in R$.
- R is antisymmetric iff for all a, b, if $(a, b) \in R$ and $a \neq b$, then $(b, a) \notin R$.
- R is transitive iff for all a, b, if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

Let R, S be relations on A. Then:

- $R \circ S = \{(a, c) : \exists b \text{ such that } (a, b) \in R \text{ and } (b, c) \in S\}$

1. Relations Examples

(a) Suppose that R, S are relations on the integers, where $R = \{(1, 2), (4, 3), (5, 5)\}$ and $S = \{(2, 5), (2, 7), (3, 3)\}$. What is $R \circ S$? What is $S \circ R$?

(b) Consider the relation $R \subseteq \mathbb{Z} \times \mathbb{Z}$ defined by $(a, b) \in R$ iff $a \leq b + 1$. List 3 pairs of integers that are in R, and 3 pairs of integers that are not.

(c) Consider the relation $R \subseteq \mathbb{Z} \times \mathbb{Z}$ defined by $(a, b) \in R$ iff $a \leq b+1$. Determine if R is reflexive, symmetric, antisymmetric, and/or transitive. If a relation has a property, explain why. If not, state a counterexample.

2. Relations Proofs

Suppose that $R, S \subseteq \mathbb{Z} \times \mathbb{Z}$ are relations.

(a) Prove or disprove: If R and S are transitive, $R \cup S$ is transitive.

(b) Prove or disprove: If R and S are reflexive, then $R \circ S$ is reflexive.

(c) Prove or disprove: If $R \circ S$ is reflexive, then R and S are reflexive.

(d) Prove or disprove: If R is symmetric, $\overline{R}$ (the complement of R) is symmetric.

3. Constructing DFAs

For each of the following, construct a DFA for the specified language.

(a) Strings of a's and b's with odd length ($\Sigma = \{a, b\}$).

(b) Strings with an even number of a's ($\Sigma = \{a, b\}$).

(c) Strings with an odd number of b's ($\Sigma = \{a, b\}$).

(d) Strings with an even number of a's or an odd number of b's ($\Sigma = \{a, b\}$).

4. Structural Induction: Dictionaries

Recursive definition of a Dictionary (i.e. a Map):

- Basis Case: [] is the empty dictionary
- Recursive Case: If D is a dictionary, and a and b are elements of the universe, then $(a \to b)$ :: D is a dictionary that maps a to b (in addition to the content of D).

Recursive functions on Dictionaries:

$$\begin{aligned} \text{AllKeys}(\texttt{[]}) &= \texttt{[]} \\ \text{AllKeys}((a \to b) :: \text{D}) &= a :: \text{AllKeys}(\text{D}) \\ \text{len}(\texttt{[]}) &= 0 \\ \text{len}((a \to b) :: \text{D}) &= 1 + \text{len}(\text{D}) \end{aligned}$$

Recursive functions on Sets:

$$\begin{aligned} \text{len}(\texttt{[]}) &= 0 \\ \text{len}(a :: \text{C}) &= 1 + \text{len}(\text{C}) \end{aligned}$$

Statement to prove:
Prove that $\text{len}(\text{D}) = \text{len}(\text{AllKeys}(\text{D}))$.

5. Structural Induction on Palindromes

Consider the following *recursive* defintion of the set B of palindrome binary strings:

- **Base case:** $\varepsilon \in B$, $0 \in B$, $1 \in B$.
- **Recursive steps**:
  - If $s \in B$, then $0s0 \in E$, $1s1 \in B$, and $ss \in B$.

  Now define the functions numOnes(x) and numZeros(x) to be the number of 1s and 0s respectively in the string x.

Use structural induction to prove that for any string $s \in B$, numOnes$(s) \cdot$ numZeros(s) is even.