CSE 390Z: Mathematics for Computation Workshop

Week 3 Workshop Problems

Conceptual Review

(a) Inference Rules:

Introduce ∨:	$\frac{A}{\therefore\ A \vee B,\ B \vee A}$	Eliminate ∨:	$\frac{A \vee B\ ;\ \neg A}{\therefore\ B}$
Introduce ∧:	$\frac{A\ ;\ B}{\therefore\ A \wedge B}$	Eliminate ∧:	$\frac{A \wedge B}{\therefore\ A\ ,\ B}$
Direct Proof:	$\frac{A \Rightarrow B}{\therefore\ A \rightarrow B}$	Modus Ponens:	$\frac{A\ ;\ A \rightarrow B}{\therefore\ B}$

(b) Given $A \wedge B$, prove $A \vee B$

Given $P \rightarrow R$, $R \rightarrow S$, prove $P \rightarrow S$.

(c) What is a predicate, a domain of discourse, and a quantifier?

(d) When translating to predicate logic, how do you restrict to a smaller domain in a "for all"? How do you restrict to a smaller domain in an "exists"?

(e) What are DeMorgan's Laws for Quantifiers?

1. Predicate Logic Translations

Let the domain of discourse be all animals. Let $\texttt{Panda}(x)$ ::= "x is a panda" and $\texttt{KungFu}(x)$ ::= x knows kung fu. Translate the following statements to English.

(a) $\exists x(\neg\,\texttt{Panda}(x) \wedge \texttt{KungFu}(x))$

(b) $\forall x(\texttt{Panda}(x) \rightarrow \texttt{KungFu}(x))$

(c) $\neg\exists y(\texttt{Panda}(y) \wedge \neg\,\texttt{KungFu}(y))$

Your friend translated the sentence "there exists a panda who knows kung fu" to $\exists x(\text{Panda}(x) \rightarrow \text{KungFu}(x))$. This is wrong! Let's understand why.

(d) Use the Law of Implications to rewrite the translation without the $\rightarrow$.

(e) Translate the predicate from (d) back to English. How does this differ from the intended meaning?

(f) This is a warning to be very careful when including an implication nested under an exists! It should almost always be avoided, unless there is a forall involved as well. (Nothing to write for this part).

2. Tricky Translations

Translate the following logical statements to natural English sentences. The domain of discourse is movies and actors. The following predicates are defined: Movie(x) ::= x is a movie, Actor(x) ::= x is an actor, Features(x, y) ::= x features y.

(a) $\neg\exists x\exists y\exists z(\text{Movie}(x) \wedge \text{Actor}(y) \wedge \text{Actor}(z) \wedge y \neq z \wedge \text{Features}(x, y) \wedge \text{Features}(x, z))$

(b) $\neg\forall x((\text{Movie}(x) \land \text{Feature}(x, \text{Daniel Radcliffe})) \to x = \text{Harry Potter})$

(c) Below are logical expressions that look very similar, but only one is a correct translation of the sentence: "There is an actor that is featured in every movie". Find the correct translation and explain why the other options are wrong/nonsensical.

$\exists y \forall x((\text{Actor}(x) \land \text{Movie}(y)) \to \text{Features}(y, x))$

$\exists x \forall y(\text{Actor}(x) \land (\text{Movie}(y) \to \text{Features}(y, x))$

$\exists y \forall x(\text{Actor}(x) \land (\text{Movie}(y) \to \text{Features}(y, x))$

$\forall x \exists y(\text{Actor}(x) \land (\text{Movie}(y) \to \text{Features}(y, x))$

3. More Tricky Translations

Express the following sentences in predicate logic. The domain of discourse is movies and actors. You may use the following predicates: $\text{Movie}(x) ::= x$ is a movie, $\text{Actor}(x) ::= x$ is an actor, $\text{Features}(x, y) ::= x$ features y.

(a) Every movie features an actor.

(b) Not every actor has been featured in a movie.

(c) All movies that feature Harry Potter must feature Voldermort.
Hint: You can use "Harry Potter" and "Voldemort" as constants that you can directly plug into a predicate.

(d) There is a movie that features **exactly one** actor.

4. Domains of Discourse

For the following, find a domain of discourse where the following statement is true and another where it is false. Note that for the arithmetic symbols to make sense, the domains of discourse should be sets of numbers.

(a) $\exists x(2x = 0)$

(b) $\forall x \exists y(x + y = 0)$

(c) $\exists x \forall y(x + y = y)$

5. Negating Quantifiers

In the previous question, we translated the sentence "Not every actor has been featured in a movie" to predicate logic.

This was Kriti's translation: $\neg\forall x(\text{Actor}(x) \to \exists y(\text{Movie}(y) \wedge \text{Features}(y, x)))$

This was Tanush's translation: $\exists x(\text{Actor}(x) \wedge \forall y(\text{Movie}(y) \to \neg\text{Features}(y, x)))$

(a) Azita claims that Kriti and Tanush are both correct. Do you agree with Azita?

(b) Use a chain of predicate logic equivalences to prove that the two translations are equivalent.
Hint: You may wish to use DeMorgan's Law for Predicates and the Law of Implication.

6. Translations with Integers

Translate the following English sentences to predicate logic. The domain is integers, and you may use $=$, $\neq$, and $>$ as predicates. Assume the predicates Prime, Composite, and Even have been defined appropriately.
Note: Composite numbers are ones that have at least 2 factors (the opposite of prime).

(a) 2 is prime.

(b) Every **positive** integer is prime or composite, but not both.

(c) There is **exactly one** even prime.

(d) 2 is the only even prime.

(e) Some, but not all, composite integers are even.

7. Formal Proofs: Modus Ponens

(a) Prove that given $p \to q$, $\neg s \to \neg q$, and p, we can conclude s.

(b) Prove that given $\neg(p \vee q) \to s$, $\neg p$, and $\neg s$, we can conclude q.

8. Formal Proofs: Direct Proof Rule

(a) Prove that given $p \to q$, we can conclude $(p \wedge r) \to q$

(b) Prove that given $p \vee q$, $q \to r$, and $r \to s$, we can conclude $\neg p \to s$.

9. Challenge: Predicate Negation

Translate "You can fool all of the people some of the time, and you can fool some of the people all of the time, but you can't fool all of the people all of the time" into predicate logic. Then, negate your translation. Then, translate the negation back into English.

Hint: Let the domain of discourse be all people and all times, and let $P(x, y)$ be the statement "You can fool person x at time y". You can get away with not defining any other predicates if you use P.