QuickCheck: Structural Induction Solutions

Please submit a response to the following questions on Gradescope. We do not grade on accuracy, so please submit your best attempt. You may either typeset your responses or hand-write them. Note that hand-written solutions must be legible to be graded.

We have created this template if you choose to typeset with Latex. This guide has specific information about scanning and uploading pdf files to Gradescope.

0. How Many Ones?
The set $T$ is defined as follows:
- Base case: $\epsilon \in T$
- Recursive Rules:
  - If $x \in T$, then $11x \in T$
  - If $x \in T$ and $y \in T$, then $x0y \in T$

Given the following recursively defined function
- $\text{numOnes}(\epsilon) = 0$
- $\text{numOnes}(11x) = 2 + \text{numOnes}(x)$
- $\text{numOnes}(x0y) = \text{numOnes}(x) + \text{numOnes}(y)$

Prove that for all strings $n$ in $T$, $\text{numOnes}(n)$ is even

Hint: In structural induction, the structure of your induction mirrors the recursive definition.

Solution:
Let $P(n)$ be "2 | $\text{numOnes}(n)$". We will show that $P(n)$ is true for all $n \in T$ by structural induction.

**Base Case** ($n = \epsilon$):
$\text{numOnes}(\epsilon) = 0$ definition of $\text{numOnes}$
$0 = 2 \cdot 0$ and $2|0$ by definition of divides.
Therefore $P(0)$ holds true.

**Induction Hypothesis:** Suppose $P(x)$ and $P(y)$ are true for some arbitrary elements $x, y \in T$.

**Induction Step:**

**Goal:** Prove $P(11x)$ and $P(x0y)$

$\text{numOnes}(11x) = 2 + \text{numOnes}(x)$ by definition of $\text{numOnes}$. By the inductive hypothesis, $2 | \text{numOnes}(x)$.
Therefore, by definition of divides $\text{numOnes}(x) = 2z$ for some integer $z$. Thus,

$\text{numOnes}(11x) = 2 + \text{numOnes}(x) = 2z + 2 = 2(z + 1)$

Therefore, by definition of divides, $2 | \text{numOnes}(11x)$. Therefore, $P(11x)$ holds.
numOnes(x0y) = numOnes(x) + numOnes(y) by definition of numOnes. By the induction hypothesis, 2 | numOnes(x) and 2 | numOnes(y). Therefore, by definition of divides, numOnes(x) = 2z for some integer z and numOnes(y) = 2q for some integer q. Thus,

numOnes(x0y) = numOnes(x) + numOnes(y) = 2z + 2q = 2(z + q)

Therefore, by definition of divides, 2 | numOnes(x0y). Therefore, P(x0y) holds.

The result follows for all n ∈ T by structural induction.

1. Video Solution
Watch this video on the solution after making an initial attempt. Then, answer the following questions.

(a) What is one thing you took away from the video solution?