CSE 390Z: Mathematics for Computation Workshop

Practice 311 Midterm Solutions

Name: ______________________________

UW ID: ______________________________

Instructions:

- This is a **simulated practice midterm**. You will **not** be graded on your performance on this exam.
- Nevertheless, please treat this as if it is a real exam. That means that you may not discuss with your neighbors, reference outside material, or use your devices during the next 50 minute period.
- If you get stuck on a problem, consider moving on and coming back later. In the actual exam, there will likely be opportunity for partial credit.
- There are 5 problems on this exam, totaling 85 points.

1. Predicate Translation [20 points]
Let the domain of discourse be numbers and mathematicians (math teachers and math students). Let the following predicates be defined:

Number(x) := x is a number
Prime(x) := x is a prime number
MathTeacher(x) := x is a math teacher
MathStudent(x) := x is a math student
Likes(x, y) := x likes y

For parts (a) and (b), translate the predicate logic statement into natural English.

(a) (5 points) $\forall x \forall y((\text{Prime}(x) \wedge \text{Likes}(y, x)) \rightarrow \text{MathStudent}(y))$

Solution:

Only math students like prime numbers.

(b) (5 points) $\exists x(\text{Prime}(x) \wedge \forall y((\text{MathTeacher}(y) \vee \text{MathStudent}(y)) \rightarrow \text{Likes}(y, x)))$

Solution:

There is a prime number that all mathematicians like.

For parts (c) and (d), translate the English sentence into predicate logic.

(c) (5 points) There are at least two (different) prime numbers

Solution:

$\exists x \exists y(\text{Prime}(x) \wedge \text{Prime}(y) \wedge x \neq y)$

(d) (5 points) Math teachers and math students don't like the same numbers.

Solution:

$\forall x \forall y \forall z((\text{MathTeacher}(x) \wedge \text{MathStudent}(y) \wedge \text{Number}(z)) \rightarrow \neg(\text{Likes}(x, z) \wedge \text{Likes}(y, z)))$
OR
$\neg \exists x \exists y \exists z(\text{Number}(x) \wedge \text{MathTeacher}(y) \wedge \text{MathStudent}(z) \wedge \text{Likes}(y, x) \wedge \text{Likes}(z, x))$

2. Boolean Algebra [15 points]

Let f be the boolean function defined as $f(x, y, z) = (x + y)' + (zy)$

(a) (5 points) Fill in the following table with the values of $f(x, y, z)$ in the last column. Feel free to use the blank columns while doing your work.

x	y	z				$f(x, y, z)$
0	0	0				
0	0	1				
0	1	0				
0	1	1				
1	0	0				
1	0	1				
1	1	0				
1	1	1				

Solution:

x	y	z	$(x+y)$	$(x+y)'$	(zy)	$f(x,y,z)$
0	0	0	0	1	0	1
0	0	1	0	1	0	1
0	1	0	1	0	0	0
0	1	1	1	0	1	1
1	0	0	1	0	0	0
1	0	1	1	0	0	0
1	1	0	1	0	0	0
1	1	1	1	0	1	1

(b) (5 points) Write $f(x,y,z)$ as a Boolean algebra expression in the canonical sum-of-products form in terms of the three input bits x, y, and z.

Solution:

$f(x,y,z) = x'y'z' + x'y'z + x'yz + xyz$

(c) (5 points) Write $f(x,y,z)$ as a Boolean algebra expression in the canonical product-of-sums form in terms of the three input bits x, y, and z.

Solution:

$f(x,y,z) = (x + y' + z)(x' + y + z)(x' + y + z')(x' + y' + z)$

3. Number Theory Proof [10 points]

Prove the following statement using a formal proof:

For all integers a, b, c, if $a^2 \mid b$ and $b^3 \mid c$, then $a^6 \mid c$.

Solution:

1. Let x, y, z be arbitrary integers.

2.1	$x^2 \mid y \wedge y^3 \mid z$	Assumption
2.2	$x^2 \mid y$	Elim $\wedge$: 2.1
2.3	$y^3 \mid z$	Elim $\wedge$: 2.2
2.4	$\exists k(y = x^2 k)$	Definition of divides: 2.2
2.5	$\exists k(z = y^3 k)$	Definition of divides: 2.3
2.6	$y = x^2 s$	Elim $\exists$: 2.4
2.7	$z = y^3 t$	Elim $\exists$: 2.5
2.8	$z = (x^2 s)^3 t = x^6 (s^3 t)$	Algebra
2.9	$\exists k(z = x^6 k)$	Intro $\exists$: 2.8
2.10	$x^6 \mid z$	Definition of divides: 2.9

2. $x^2 \mid y \wedge y^3 \mid z \rightarrow x^6 \mid z$ — Direct Proof: 2.1 - 2.10
3. $\forall a \forall b \forall c((a^2 \mid b \wedge b^3 \mid c) \rightarrow a^6 \mid c$ — Intro $\forall$

4. Induction [20 points]

Prove the following using induction:

$$\text{For all integers } n \geq 0, \ \sum_{j=1}^{n}(2j-1) = n^2$$

Hint: When the lower bound of a summation is greater than the upper bound, the summation is 0. For example, $\sum_{i=0}^{-1} = 0$.

Solution:

1. Let $P(n)$ be $\sum_{j=1}^{n}(2j-1) = n^2$. We will prove $P(n)$ true for all integers $n \geq 0$ by induction.

2. Base Case ($n = 0$):
$\sum_{j=1}^{0}(2j-1) = 0 = 0^2$. So, $P(0)$ holds.

3. Inductive Hypothesis: Assume that $P(k)$ holds for some arbitrary integer $k \geq 0$; i.e, $\sum_{j=1}^{k}(2j-1) = k^2$.

4. Inductive Step:

> Goal: Show $P(k+1)$, i.e. show $\sum_{j=1}^{k+1}(2j-1) = (k+1)^2$

$$\begin{aligned}
\sum_{j=1}^{k+1}(2j-1) &= (\sum_{j=1}^{k}(2j-1)) + (2(k+1)-1) \\
&= k^2 + (2(k+1)-1) && \text{Inductive Hypothesis} \\
&= k^2 + 2k + 1 \\
&= (k+1)^2
\end{aligned}$$

Thus, $P(k+1)$ holds.

5. Therefore, by the principle of induction, the claim $P(n)$ holds for all integers $n \geq 0$.

5. Sets [20 points]

One of the following claims is true while the other is false. For the true claim, prove it with an English proof. For the false claim, disprove it with a counterexample (you can write elements for sets A, B, C and show that it disproves the claim).

For both parts, suppose A, B, and C are sets.

(a) (10 points) If $A \subseteq B$, then $A \setminus C \subseteq B \setminus C$.

Solution:

This claim is true. The proof is as follows:

Suppose $A \subseteq B$. Let x be an arbitrary element in $A \setminus C$. By definition of set difference, $x \in A$ and $x \notin C$. By definition of subset, $x \in B$. Because $x \in B$ and $x \notin C$, $x \in B \setminus C$ by definition of set difference. Since x was arbitrary, we have shown by definition of subset that $A \setminus C \subseteq B \setminus C$. Thus, the claim holds.

(b) (10 points) If $B \subseteq C$, then $A \setminus B \subseteq A \setminus C$.

Solution:

This claim is false. Let

$A = \{1, 2, 3\}$

$B = \{1\}$

$C = \{1, 2\}$

Then, we can see that $B \subseteq C$. $A \setminus B = \{2, 3\}$ and $A \setminus C = \{3\}$. Since the element 2 is in $A \setminus B$ but not in $A \setminus C$, $A \setminus B \nsubseteq A \setminus C$ and the claim is false.