CSE 390Z: Mathematics for Computation Workshop

Practice 311 Midterm Solutions

Name: ____________________________

UW ID: __________________________

Instructions:

- This is a simulated practice midterm. You will not be graded on your performance on this exam.

- Nevertheless, please treat this as if it is a real exam. That means that you may not discuss with your neighbors, reference outside material, or use your devices during the next 50 minute period.

- If you get stuck on a problem, consider moving on and coming back later. In the actual exam, there will likely be opportunity for partial credit.

- There are 5 problems on this exam, totaling 85 points.
1. **Predicate Translation** [20 points]

Let the domain of discourse be numbers and mathematicians (math teachers and math students). Let the following predicates be defined:

\[ \text{Number}(x) := x \text{ is a number} \]
\[ \text{Prime}(x) := x \text{ is a prime number} \]
\[ \text{MathTeacher}(x) := x \text{ is a math teacher} \]
\[ \text{MathStudent}(x) := x \text{ is a math student} \]
\[ \text{Likes}(x, y) := x \text{ likes } y \]

For parts (a) and (b), translate the predicate logic statement into natural English.

(a) (5 points) \( \forall x \forall y ((\text{Prime}(x) \land \text{Likes}(y, x)) \to \text{MathStudent}(y)) \)

**Solution:** 
Only math students like prime numbers.

(b) (5 points) \( \exists x (\text{Prime}(x) \land \forall y ((\text{MathTeacher}(y) \lor \text{MathStudent}(y)) \to \text{Likes}(y, x))) \)

**Solution:** 
There is a prime number that all mathematicians like.

For parts (c) and (d), translate the English sentence into predicate logic.

(c) (5 points) There are at least two (different) prime numbers

**Solution:**
\[ \exists x \exists y (\text{Prime}(x) \land \text{Prime}(y) \land x \neq y) \]

(d) (5 points) Math teachers and math students don’t like the same numbers.

**Solution:**
\[ \forall x \forall y \forall z ((\text{MathTeacher}(x) \land \text{MathStudent}(y) \land \text{Number}(z)) \to \neg ((\text{Likes}(x, z) \land \text{Likes}(y, z)))) \]

OR
\[ \neg \exists x \exists y \exists z (\text{Number}(x) \land \text{MathTeacher}(y) \land \text{MathStudent}(z) \land \text{Likes}(y, x) \land \text{Likes}(z, x)) \]
2. Boolean Algebra [15 points]

Let $f$ be the boolean function defined as $f(x, y, z) = (x + y)' + (zy)$

(a) (5 points) Fill in the following table with the values of $f(x, y, z)$ in the last column. Feel free to use the blank columns while doing your work.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$f(x, y, z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
Solution:

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$y$</td>
<td>$z$</td>
<td>$(x + y)$</td>
<td>$(x + y)'$</td>
<td>$(zy)$</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

(b) (5 points) Write $f(x, y, z)$ as a Boolean algebra expression in the canonical sum-of-products form in terms of the three input bits $x, y,$ and $z$.

Solution:

$$f(x, y, z) = x'y'z' + x'y'z + x'yz + xyz$$

(c) (5 points) Write $f(x, y, z)$ as a Boolean algebra expression in the canonical product-of-sums form in terms of the three input bits $x, y,$ and $z$.

Solution:

$$f(x, y, z) = (x + y' + z)(x' + y + z')(x' + y' + z)$$
3. **Number Theory Proof** [10 points]

Prove the following statement using a formal proof:

For all integers $a, b, c$, if $a^2 \mid b$ and $b^3 \mid c$, then $a^6 \mid c$.

**Solution:**

1. Let $x, y, z$ be arbitrary integers.

   2.1 $x^2 \mid y \land y^3 \mid z$ \hspace{1cm} Assumption

   2.2 $x^2 \mid y$ \hspace{1cm} Elim $\land$: 2.1

   2.3 $y^3 \mid z$ \hspace{1cm} Elim $\land$: 2.2

   2.4 $\exists k (y = x^2 k)$ \hspace{1cm} Definition of divides: 2.2

   2.5 $\exists k (z = y^3 k)$ \hspace{1cm} Definition of divides: 2.3

   2.6 $y = x^2 s$ \hspace{1cm} Elim $\exists$: 2.4

   2.7 $z = y^3 t$ \hspace{1cm} Elim $\exists$: 2.5

   2.8 $z = (x^2 s)^3 t = x^6 (s^3 t)$ \hspace{1cm} Algebra

   2.9 $\exists k (z = x^6 k)$ \hspace{1cm} Intro $\exists$: 2.8

   2.10 $x^6 \mid z$ \hspace{1cm} Definition of divides: 2.9

2. $x^2 \mid y \land y^3 \mid z \rightarrow x^6 \mid z$ \hspace{1cm} Direct Proof: 2.1 - 2.10

3. $\forall a \forall b \forall c ((a^2 \mid b \land b^3 \mid c) \rightarrow a^6 \mid c)$ \hspace{1cm} Intro $\forall$
4. **Induction** [20 points]

Prove the following using induction:

\[ \sum_{j=1}^{n} (2j - 1) = n^2 \]

*Hint: When the lower bound of a summation is greater than the upper bound, the summation is 0. For example, \( \sum_{i=0}^{-1} \) is 0.*

**Solution:**

1. Let \( P(n) \) be \( \sum_{j=1}^{n} (2j - 1) = n^2 \). We will prove \( P(n) \) true for all integers \( n \geq 0 \) by induction.

2. **Base Case (\( n = 0 \)):**
   \[ \sum_{j=1}^{0} (2j - 1) = 0 = 0^2 \]. So, \( P(0) \) holds.

3. **Inductive Hypothesis:** Assume that \( P(k) \) holds for some arbitrary integer \( k \geq 0 \); i.e., \( \sum_{j=1}^{k} (2j - 1) = k^2 \).

4. **Inductive Step:**

   \[
   \begin{align*}
   \sum_{j=1}^{k+1} (2j - 1) &= (\sum_{j=1}^{k} (2j - 1)) + (2(k+1) - 1) \\
   &= k^2 + (2(k + 1) - 1) \\
   &= k^2 + 2k + 1 \\
   &= (k + 1)^2 
   \end{align*}
   \]

   Thus, \( P(k+1) \) holds.

5. Therefore, by the principle of induction, the claim \( P(n) \) holds for all integers \( n \geq 0 \).
5. Sets [20 points]

One of the following claims is true while the other is false. For the true claim, prove it with an English proof. For the false claim, disprove it with a counterexample (you can write elements for sets $A, B, C$ and show that it disproves the claim).

For both parts, suppose $A, B,$ and $C$ are sets.

(a) (10 points) If $A \subseteq B$, then $A \setminus C \subseteq B \setminus C$.

**Solution:**
This claim is true. The proof is as follows:
Suppose $A \subseteq B$. Let $x$ be an arbitrary element in $A \setminus C$. By definition of set difference, $x \in A$ and $x \notin C$. By definition of subset, $x \in B$. Because $x \in B$ and $x \notin C$, $x \in B \setminus C$ by definition of set difference. Since $x$ was arbitrary, we have shown by definition of subset that $A \setminus C \subseteq B \setminus C$. Thus, the claim holds.

(b) (10 points) If $B \subseteq C$, then $A \setminus B \subseteq A \setminus C$.

**Solution:**
This claim is false. Let
$A = \{1, 2, 3\}$
$B = \{1\}$
$C = \{1, 2\}$
Then, we can see that $B \subseteq C$. $A \setminus B = \{2, 3\}$ and $A \setminus C = \{3\}$. Since the element 2 is in $A \setminus B$ but not in $A \setminus C$, $A \setminus B \nsubseteq A \setminus C$ and the claim is false.