CSE 390Z: Mathematics for Computation Workshop

Practice 311 Final Solutions

Name: ______________________________

UW ID: ______________________________

Instructions:

- This is a **simulated practice final**. You will **not** be graded on your performance on this exam.
- Nevertheless, please treat this as if it is a real exam. That means that you may not discuss with your neighbors, reference outside material, or use your devices during the next 50 minute period.
- If you get stuck on a problem, consider moving on and coming back later. In the actual exam, there will likely be opportunity for partial credit.
- There are 7 problems on this exam, totaling 120 points.

1. Language Representation (18 points)

Let the language L consist of all binary strings that do not contain 111.

(a) [5 points] Write a regular expression that represents L.

Solution:

$(0 \cup 10 \cup 110)^*(1 \cup 11 \cup \epsilon)$

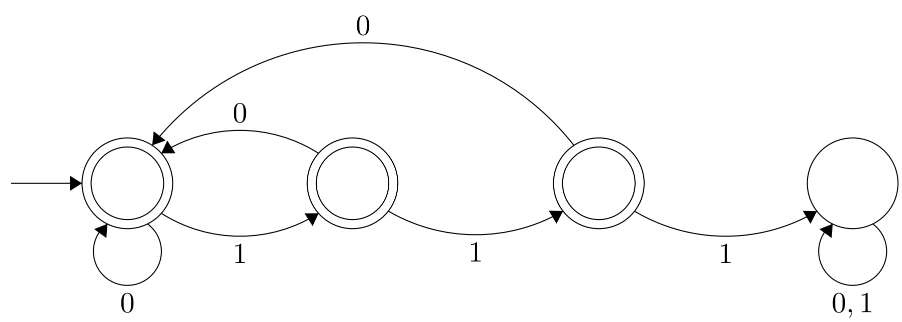
(b) [5 points] Write a CFG that generates all strings in L.

Solution:

S $\rightarrow$ **XY**
X $\rightarrow 0$**X** $\mid 10$**X** $\mid 110$**X** $\mid \epsilon$
Y $\rightarrow 1 \mid 11 \mid \epsilon$

(c) [5 points] Draw a DFA that accepts exactly the strings in L.

Solution:



(d) [3 points] Convert the following regular expression to a CFG:

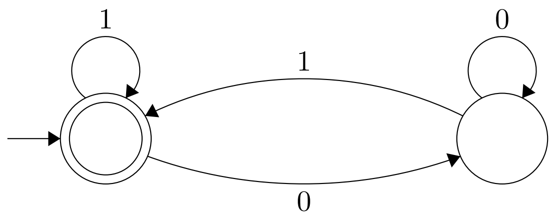
$$10(0^* \cup 1^*)01$$

Solution:

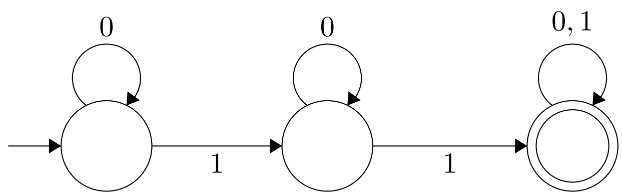
S $\rightarrow 10$**X**$01 \mid 10$**Y**01
X $\rightarrow 0$**X** $\mid \epsilon$
Y $\rightarrow 1$**Y** $\mid \epsilon$

2. FSMs (20 points)

(a) [10 points] The following DFA accepts all binary strings that do not end in 0.

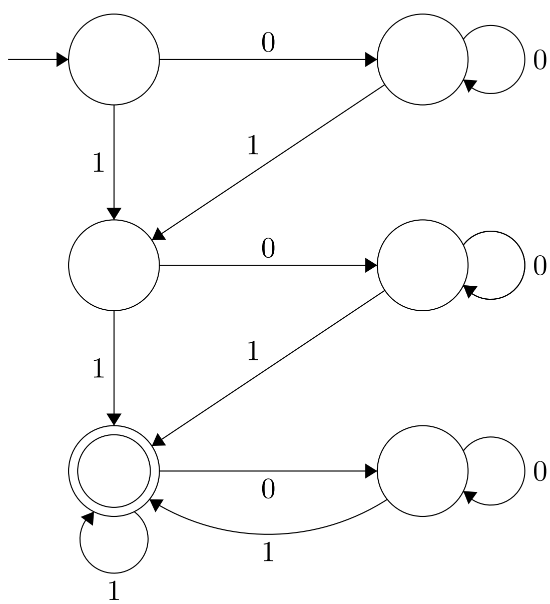


The following DFA accepts all binary strings that contain at least two 1s.

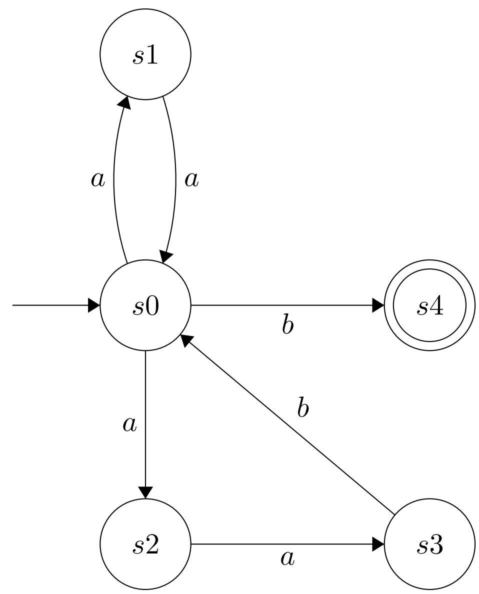


Draw a DFA that accepts all binary strings that do not end in 0 AND contain at least two 1s.

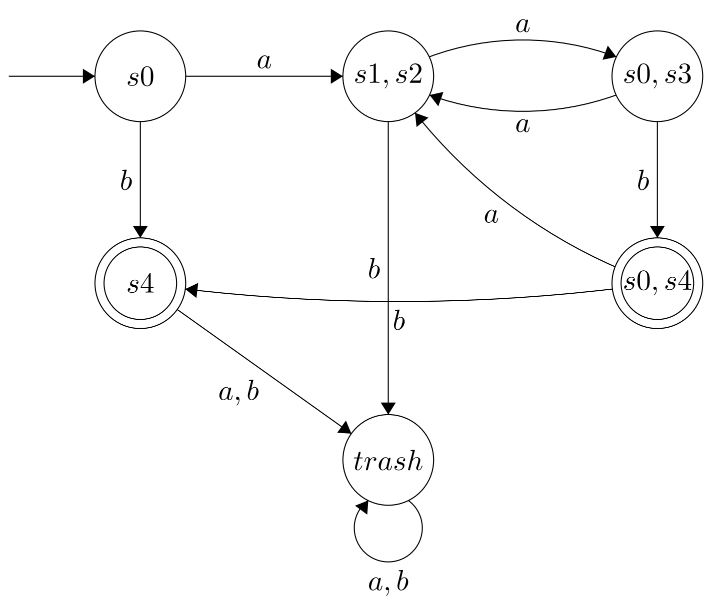
Solution:



(b) [10 points] Convert the following NFA into a DFA.



Solution:



3. Relations (17 points)

(a) [7 points] Let the relation $R \subseteq \mathbb{Z} \times \mathbb{Z}$ be defined as $\{(x, y) : 4|(x + y)\}$.
List 3 elements of R.

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-
-

Solution:

- (0,0)
- (4,0)
- (2,2)

List the properties that R has out of the following: reflexive, transitive, symmetric, antisymmetric. If R has a property, simply say so without further explanation. If R does not have a property, provide a counterexample without explaining further.

Solution:

Reflexive: No. (1,1) is not in R.
Transitive: No. (3,1) and (1,3) are in R, but (3,3) is not.
Symmetric: Yes.
Antisymmetric: No. (3,1) and (1,3) are both in R.

(b) [10 points] Let R and S be symmetric relations on a set A. Prove that $R \setminus S$ is symmetric using an *English proof*.

Solution:

Let a, b be arbitrary and suppose $(a, b) \in R \setminus S$. By definition of set difference, $(a, b) \in R$ and $(a, b) \notin S$. Since R is symmetric, $(b, a) \in R$. By taking the contrapositive of the definition of symmetric, we see that if an element $(x, y) \notin S$, then $(y, x) \notin S$ for any x, y. So, $(b, a) \notin S$. By definition of set difference, $(b, a) \in R \setminus S$. Since a, b were arbitrary, we have shown that $R \setminus S$ is symmetric.

4. Induction I (15 points)

Let the function $f : \mathbb{N} \to \mathbb{N}$ be defined as follows:

$f(0) = 2$
$f(1) = 7$
$f(n) = f(n-1) + 2f(n-2)$ for $n \geq 2$
Prove that $f(n) = 3 * 2^n + (-1)^{n+1}$ for all integers $n \geq 0$ using strong induction.

Solution:
Let $P(n)$ be "$f(n) = 3 * 2^n + (-1)^{n+1}$. We will prove $P(n)$ holds for all $n \geq 0$ by strong induction.

Base Cases:
$n = 0$:
$f(0) = 2$
$3 * 2^0 + (-1)^{0+1} = 3 * 1 + (-1)^1 = 3 + (-1) = 2.$
$2 = 2$, so $P(0)$ holds.

$n = 1$:
$f(1) = 7$
$3 * 2^1 + (-1)^{1+1} = 3 * 2 + (-1)^2 = 6 + 1 = 7$
$7 = 7$, so $P(1)$ holds.

Inductive Hypothesis: Suppose that $P(j)$ holds for all $0 \leq j \leq k$ for some arbitrary integer k.
Inductive Step:

Goal: Show $P(k+1)$, i.e. show $f(k+1) = 3 * 2^{k+1} + (-1)^{(k+1)+1}$

$$
\begin{aligned}
f(k+1) &= f((k+1)-1) + 2f((k+1)-2) && \text{def. of } f \\
&= f(k) + 2f(k-1) \\
&= 3 * 2^k + (-1)^{k+1} + 2(3 * 2^{k-1} + (-1)^{(k-1)+1}) && \text{IH} \\
&= 3 * 2^k + (-1)^{k+1} + 2(3 * 2^{k-1} + (-1)^k) \\
&= 3 * 2^k + (-1)^{k+1} + 3 * 2^k + 2(-1)^k \\
&= 2 * 3 * 2^k + (-1)^{k+1} + 2(-1)^k \\
&= 3 * 2^{k+1} + (-1)^{k+1} + 2(-1)^k \\
&= 3 * 2^{k+1} + (-1)(-1)(-1)^{k+1} + 2(-1)(-1)(-1)^k && (-1)(-1) = 1 \\
&= 3 * 2^{k+1} - (-1)^{k+2} + 2(-1)^{k+2} \\
&= 3 * 2^{k+1} + (-1)^{(k+1)+1}
\end{aligned}
$$

Thus, $P(k+1)$ holds.

We conclude that $P(n)$ holds for all $n \geq 0$ by strong induction.

5. Induction II (15 points)

Let the set S be recursively defined as follows:
Basis: $(0, 0) \in S$
Recursive Step: If $(x, y) \in S$, then $(x + 2, y + 4) \in S$ and $(x + 4, y + 8) \in S$.

Prove that for all $(x, y) \in S$, $x + y$ is divisible by 3.

Solution:
Define $P((x, y))$ to be the claim $3 \mid (x + y)$. We will prove that $P((x, y))$ holds for all $(x, y) \in S$ by structural induction.
Base Case: $(x, y) = (0, 0)$
$0 + 0 = 0 = 3 * 0$. So, $3|(x + y)$ and $P((0, 0))$ holds.

Inductive Hypothesis: Suppose that $P((a, b))$ holds for some arbitrary $(a, b) \in S$. (i.e. $3|(a + b)$).

Inductive Step:

Goal: Show $P((a+2, b+4))$ and $P((a+4, b+8))$

By the inductive hypothesis, $3 \mid (a + b)$. By definition of divides, $3k = a + b$ for some integer k.

$$(a + 2) + (b + 4) = a + b + 6 = 3k + 6 = 3(k + 2)$$

So, $3 \mid ((a + 2) + (b + 4))$ which means $P((a + 2, b + 4))$ holds.

$$(a + 4) + (b + 8) = a + b + 12 = 3k + 12 = 3(k + 4)$$

So, $3 \mid ((a + 4) + (b + 8))$ which means $P((a + 4, b + 8))$ holds.

Thus, $P((x, y))$ holds for all $(x, y) \in S$ by the principle of structural induction.

6. Sets (20 points)

(a) [12 points] Prove using the Meta Theorem that for any sets A and B,

$$A \setminus \overline{B} = (A \cup \varnothing) \cap B$$

Hint 1: The empty set, $\varnothing$, is the set that contains no elements. i.e. $\varnothing ::= \{x : F\}$.
Hint 2: If you get stuck, try working backwards!

Solution:

The claim is equivalent to $\forall x(x \in (A \setminus \overline{B}) \leftrightarrow x \in ((A \cup \varnothing) \cap B))$.
Let x be arbitrary.

$$\begin{aligned}
x \in (A \setminus \overline{B}) &\equiv x \in A \wedge \neg(x \in \overline{B}) && \text{Def of set difference} \\
&\equiv x \in A \wedge \neg\neg(x \in B) && \text{Def of complement} \\
&\equiv x \in A \wedge x \in B && \text{Double Negation} \\
&\equiv (x \in A \vee F) \wedge x \in B && \text{Identity} \\
&\equiv (x \in A \vee x \in \varnothing) \wedge x \in B && \text{Def of } \varnothing \\
&\equiv x \in (A \cup \varnothing) \wedge x \in B && \text{Def of union} \\
&\equiv x \in (A \cup \varnothing) \cap B && \text{Def of intersection}
\end{aligned}$$

Since x was arbitrary, we have shown that these sets contain the same elements and are therefore equal.

Let $A = \{\{1\}, \{1, 2, 3\}\}$ and $B = \{\{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$.

(b) [4 points] What is $A \cup B$?

Solution:

$A \cup B = \{\{1\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$

(c) [4 points] What is $A \setminus B$?

Solution:

$A \setminus B = \{\{1\}\}$

7. Irregularity (15 points)
Prove that the language $L = \{10^x 10^{x+1} 1 : x \geq 0\}$ is not regular.

Solution:
Suppose for the sake of contradiction there exists a DFA D that accepts L.

Let $S = \{10^n 1 : n \geq 0\}$. Since S contains infinitely number strings and D has a finite number of states, two strings in S must end up in the same state of D. Say those strings are $10^i 1$ and $10^j 1$ for some $i, j \geq 0$ where $i \neq j$. Now, append $0^{i+1} 1$ to both strings. The resulting strings are:

$x = 10^i 10^{i+1} 1$. Note that $x \in L$.
$y = 10^j 10^{i+1} 1$. Note that $y \notin L$ since $i \neq j, j + 1 \neq i + 1$.

Both x and y must end up at the same state, but since $x \in L$ and $y \notin L$, that state must be both an accept state and a reject state. This is a contradiction, which means there does not exist a DFA D which accepts L. This means L is not regular.