CSE 390Z: Mathematics for Computation Workshop

Week 6 Workshop Solutions

0. Weak Induction Warmup

Prove by induction on n that for all integers $n \geq 4$, the inequality $n! > 2^n$ is true.

Complete the induction proof below.

Solution:

Let $P(n)$ be "$n! > 2^n$". We will prove $P(n)$ is true for all $n \in \mathbb{N}$, $n \geq 4$, by induction.

Base Case: (n = 4)**:** $4! = 24$ and $2^4 = 16$, since $24 > 16$, $P(4)$ is true.

Inductive Hypothesis: Suppose that $P(k)$ is true for some arbitrary integer $k \in \mathbb{N}$, $k \geq 4$.

Inductive Step:

$$\boxed{\text{Goal: Show } P(k+1)\text{, i.e. show } (k+1)! > 2^{k+1}}$$

$$\begin{aligned}
(k+1)! &= k! \cdot (k+1) \\
&> 2^k \cdot (k+1) && \text{(By I.H., } k! > 2^k\text{)} \\
&> 2^k \cdot 2 && \text{(Since } k \geq 4\text{, so } k+1 \geq 5 > 2\text{)} \\
&= 2^{k+1}
\end{aligned}$$

Conclusion: So by induction, $P(n)$ is true for all $n \in \mathbb{N}$, $n \geq 4$.

1. More Weak Induction

Prove that $2^n + 1 \leq 3^n$ for all positive integers n.

Solution:

Let $P(n)$ be "$2^n + 1 \leq 3^n$". We will prove that $P(n)$ holds for all integers $n \leq 1$ using induction.

Base Case: ($n = 1$):

$2^1 + 1 = 2 + 1 = 3$

$3^1 = 3$

$3 \leq 3$, so $P(1)$ holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary integer $k \geq 1$.

Inductive Step:

$$\boxed{\textbf{Goal: } \text{Show } P(k+1)\text{, i.e. } 2^{k+1} + 1 \leq 3^{k+1}.}$$

$$\begin{aligned}
2^{k+1} + 1 &= 2 * 2^k + 1 \\
&< 2 * 2^k + 2 && \text{Since } 1 < 2 \\
&= 2(2^k + 1) \\
&\leq 2 * 3^k && \text{IH} \\
&< 3 * 3^k && \text{Since } 2 < 3 \\
&= 3^{k+1}
\end{aligned}$$

So, $P(k+1)$ holds.

Conclusion: Therefore, by the principle of induction, $P(n)$ holds for all positive integers n.

2. Induction with Divides

Prove that $9 \mid (n^3 + (n+1)^3 + (n+2)^3)$ for all $n > 1$ by induction.

Solution:

Let $P(n)$ be "$9 \mid n^3 + (n+1)^3 + (n+2)^3$". We will prove $P(n)$ for all integers $n > 1$ by induction.

Base Case $(n = 2)$**:** $2^3 + (2+1)^3 + (2+2)^3 = 8 + 27 + 64 = 99 = 9 \cdot 11$, so $9 \mid 2^3 + (2+1)^3 + (2+2)^3$, so $P(2)$ holds.

Inductive Hypothesis: Assume that $9 \mid k^3 + (k+1)^3 + (k+2)^3$ for an arbitrary integer $k > 1$. Note that this is equivalent to assuming that $k^3 + (k+1)^3 + (k+2)^3 = 9j$ for some integer j by the definition of divides.

Inductive Step: $\boxed{\text{Goal: Show } 9 \mid (k+1)^3 + (k+2)^3 + (k+3)^3}$

$$\begin{aligned}
(k+1)^3 + (k+2)^3 + (k+3)^3 &= (k^2 + 6k + 9)(k+3) + (k+1)^3 + (k+2)^3 && \text{[expanding trinomial]} \\
&= (k^3 + 6k^2 + 9k + 3k^2 + 18k + 27) + (k+1)^3 + (k+2)^3 && \text{[expanding binomial]} \\
&= 9k^2 + 27k + 27 + k^3 + (k+1)^3 + (k+2)^3 && \text{[adding like terms]} \\
&= 9k^2 + 27k + 27 + 9j && \text{[by I.H.]} \\
&= 9(k^2 + 3k + 3 + j) && \text{[factoring out 9]}
\end{aligned}$$

Since k and j are integers, $k^2 + 3k + 3 + j$ is also an integer. Therefore, by the definition of divides, $9 \mid (k+1)^3 + (k+2)^3 + (k+3)^3$, so $P(k) \to P(k+1)$ for an arbitrary integer $k > 1$.

Conclusion: $P(n)$ holds for all integers $n > 1$ by induction.

3. Inductively Odd

An 123 student learning recursion wrote a recursive Java method to determine if a number is odd or not, and needs your help proving that it is correct.

```
public static boolean oddr(int n) {
   if (n == 0)
      return False;
   else
      return !oddr(n-1);
}
```

Help the student by writing an inductive proof to prove that for all integers $n \geq 0$, the method `oddr` returns `True` if n is an odd number, and `False` if n is not an odd number (i.e. n is even). You may recall the definitions Odd$(n) := \exists x \in \mathbb{Z}(n = 2x + 1)$ and Even$(n) := \exists x \in \mathbb{Z}(n = 2x)$; `!True = False` and `!False = True`.

Solution:

Let P(n) be "`oddr(n)` returns True if n is odd, or False if n is even". We will show that P(n) is true for all integers $n \geq 0$ by induction on n.

Base Case: (n $= \underline{0}$)
0 is even, so P(0) is true if `oddr(0)` returns False, which is exactly the base case of `oddr`, so P(0) is true.
Inductive Hypothesis: Suppose P(k) is true for an arbitrary integer $k \geq 0$.
Inductive Step:

- **Case 1:** $k + 1$ is even.

  If $k+1$ is even, then there is an integer x s.t. $k+1 = 2x$, so then $k = 2x-1 = 2(x-1)+1$, so therefore k is odd. We know that since $k+1 > 0$, `oddr(k+1)` should return `!oddr(k)`. By the Inductive Hypothesis, we know that since k is odd, `oddr(k)` returns True, so `oddr(k+1)` returns `!oddr(k)= False`, and $k+1$ is even, therefore P(k+1) is true.

- **Case 2:** $k + 1$ is odd.

  If $k + 1$ is odd, then there is an integer x s.t. $k + 1 = 2x + 1$, so then $k = 2x$ and therefore k is even. We know that since $k + 1 > 0$, `oddr(k+1)` should return `!oddr(k)`. By the Inductive Hypothesis, we know that since k is even, `oddr(k)` returns False, so `oddr(k+1)` returns `!oddr(k)= True`, and $k + 1$ is odd, therefore P(k+1) is true.

Then P$(k + 1)$ is true for all cases. Thus, we have shown P(n) is true for all integers $n \geq 0$ by induction.

4. Strong Induction: Recursively Defined Functions

Consider the function $f(n)$ defined for integers $n \geq 1$ as follows:
$f(1) = 1$ for $n = 1$
$f(2) = 4$ for $n = 2$
$f(3) = 9$ for $n = 3$
$f(n) = f(n-1) - f(n-2) + f(n-3) + 2(2n-3)$ for $n \geq 4$

Prove by strong induction that for all $n \geq 1$, $f(n) = n^2$.

Complete the induction proof below.

Solution:

1 Let P(n) be defined as " $f(n) = n^2$". We will prove $P(n)$ is true for all integers $n \geq 1$ by strong induction.

2 **Base Cases** ($n = 1, 2, 3$)**:**

- $n = 1$: $f(1) = 1 = 1^2$.
- $n = 2$: $f(2) = 4 = 2^2$.
- $n = 3$: $f(3) = 9 = 3^2$

So the base cases hold.

3 **Inductive Hypothesis:** Suppose for some arbitrary integer $k \geq 3$, P(j) is true for $1 \leq j \leq k$.

4 **Inductive Step:**

$\boxed{\textbf{Goal: } \text{Show } P(k+1), \text{ i.e. show that } f(k+1) = (k+1)^2.}$

$$\begin{aligned}
f(k+1) &= f(k+1-1) - f(k+1-2) + f(k+1-3) + 2(2(k+1)-3) && \text{Definition of f} \\
&= f(k) - f(k-1) + f(k-2) + 2(2k-1) \\
&= k^2 - (k-1)^2 + (k-2)^2 + 2(2k-1) && \text{By IH} \\
&= k^2 - (k^2 - 2k + 1) + (k^2 - 4k + 4) + 4k - 2 \\
&= (k^2 - k^2 + k^2) + (2k - 4k + 4k) + (-1 + 4 - 2) \\
&= k^2 + 2k + 1 \\
&= (k+1)^2
\end{aligned}$$

So P$(k+1)$ holds.

5 **Conclusion:** So by strong induction, P(n) is true for all integers $n \geq 1$.

5. Strong Induction: A Variation of the Stamp Problem

A store sells candy in packs of 4 and packs of 7. Let P(n) be defined as "You are able to buy n packs of candy". For example, $P(3)$ is not true, because you cannot buy exactly 3 packs of candy from the store. However, it turns out that P(n) is true for any $n \geq 18$. Use strong induction on n to prove this.

Hint: you'll need multiple base cases for this - think about how many steps back you need to go for your inductive step.

Solution:

Let P(n) be defined as "You are able to buy n packs of candy". We will prove $P(n)$ is true for all integers $n \geq 18$ by strong induction.

Base Cases: $(n = 18, 19, 20, 21)$**:**

- $n = 18$: 18 packs of candy can be made up of 2 packs of 7 and 1 pack of 4 $(18 = 2 * 7 + 1 * 4)$.
- $n = 19$: 19 packs of candy can be made up of 1 pack of 7 and 3 packs of 4 $(19 = 1 * 7 + 3 * 4)$.
- $n = 20$: 20 packs of candy can be made up of 5 packs of 4 $(20 = 5 * 4)$.
- $n = 21$: 21 packs of candy can be made up of 3 packs of 7 $(21 = 3 * 7)$.

Inductive Hypothesis: Suppose for some arbitrary integer $k \geq 21$, P(18) $\wedge$... $\wedge$P(k) hold.

Inductive Step:

Goal: Show $P(k + 1)$, i.e. show that we can buy $k + 1$ packs of candy.

We want to buy $k+1$ packs of candy. By the I.H., we can buy exactly $k-3$ packs, so we can add another pack of 4 packs in order to buy $k + 1$ packs of candy, so P$(k + 1)$ is true.

Note: How did we decide how many base cases to have? Well, we wanted to be able to assume P$(k-3)$, and add 4 to achieve P$(k + 1)$. Therefore we needed to be able to assume that $k - 3 \geq 18$. Adding 3 to both sides, we needed to be able to assume that $k \geq 21$. So, we have to prove the base cases up to 21, that is: $18, 19, 20, 21$.

Another way to think about this is that we had to use a fact from 4 steps back from $k + 1$ to $k - 3$ in the IS, so we needed 4 base cases.

Conclusion: So by strong induction, P(n) is true for all integers $n \geq 18$.

6. Structural Induction: Divisible by 4

Define a set $\mathfrak{B}$ of numbers by:

- 4 and 12 are in $\mathfrak{B}$
- If $x \in \mathfrak{B}$ and $y \in \mathfrak{B}$, then $x + y \in \mathfrak{B}$ and $x - y \in \mathfrak{B}$

Prove by induction that every number in $\mathfrak{B}$ is divisible by 4.
Complete the proof below:

Solution:

Let $P(b)$ be the claim that $4 \mid b$. We will prove $P(b)$ is true for all numbers $b \in \mathfrak{B}$ by structural induction.
Base Case:

- $4 \mid 4$ is trivially true, so $P(4)$ holds.
- $12 = 3 \cdot 4$, so $4 \mid 12$ and $P(12)$ holds.

Inductive Hypothesis: Suppose $P(x)$ and $P(y)$ for some arbitrary $x, y \in \mathfrak{B}$.
Inductive Step:

Goal: Prove $P(x+y)$ and $P(x-y)$

Per the IH, $4 \mid x$ and $4 \mid y$. By the definition of divides, $x = 4k$ and $y = 4j$ for some integers k, j. Then, $x + y = 4k + 4j = 4(k + j)$. Since integers are closed under addition, $k + j$ is an integer, so $4 \mid x + y$ and $P(x + y)$ holds.
Similarly, $x - y = 4k - 4j = 4(k - j) = 4(k + (-1 \cdot j))$. Since integers are closed under addition and multiplication, and -1 is an integer, we see that $k - j$ must be an integer. Therefore, by the definition of divides, $4 \mid x - y$ and $P(x - y)$ holds.
So, $P(t)$ holds in both cases.
Conclusion: Therefore, $P(b)$ holds for all numbers $b \in \mathfrak{B}$.

7. Structural Induction: CharTrees

Recursive Definition of CharTrees:

- Basis Step: $\texttt{Null}$ is a **CharTree**
- Recursive Step: If L, R are **CharTree**s and $c \in \Sigma$, then $\texttt{CharTree}(L, c, R)$ is also a **CharTree**

Intuitively, a **CharTree** is a tree where the non-null nodes store a `char` data element.

Recursive functions on CharTrees:

- The preorder function returns the preorder traversal of all elements in a **CharTree**.

$$\begin{aligned} \text{preorder}(\texttt{Null}) &= \varepsilon \\ \text{preorder}(\texttt{CharTree}(L, c, R)) &= c \cdot \text{preorder}(L) \cdot \text{preorder}(R) \end{aligned}$$

- The postorder function returns the postorder traversal of all elements in a **CharTree**.

$$\begin{aligned} \text{postorder}(\texttt{Null}) &= \varepsilon \\ \text{postorder}(\texttt{CharTree}(L, c, R)) &= \text{postorder}(L) \cdot \text{postorder}(R) \cdot c \end{aligned}$$

- The mirror function produces the mirror image of a **CharTree**.

$$\begin{aligned} \text{mirror}(\texttt{Null}) &= \texttt{Null} \\ \text{mirror}(\texttt{CharTree}(L, c, R)) &= \texttt{CharTree}(\text{mirror}(R), c, \text{mirror}(L)) \end{aligned}$$

- Finally, for all strings x, let the "reversal" of x (in symbols x^R) produce the string in reverse order.

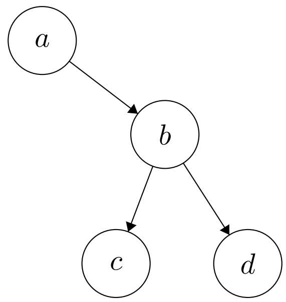
Additional Facts:
You may use the following facts:

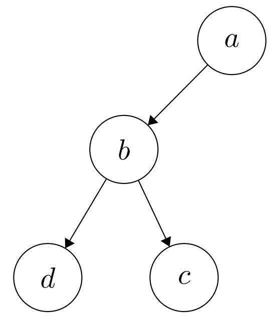
- For any strings $x_1, ..., x_k$: $(x_1 \cdot ... \cdot x_k)^R = x_k^R \cdot ... \cdot x_1^R$
- For any character c, $c^R = c$

Statement to Prove:
Show that for every **CharTree** T, the reversal of the preorder traversal of T is the same as the postorder traversal of the mirror of T. In notation, you should prove that for every **CharTree**, T: $[\text{preorder}(T)]^R = \text{postorder}(\text{mirror}(T))$.

There is an example and space to work on the next page.

Example for Intuition:





Let T_i be the tree above.
$(T_i) =$"abcd".
T_i is built as $(\texttt{null}, a, U)$
Where U is (V, b, W),
$V = (\texttt{null}, c, \texttt{null}), W = (\texttt{null}, d, \texttt{null})$.

This tree is (T_i).
$((T_i)) =$"dcba",
"dcba" is the reversal of "abcd" so
[preorder$(T_i)]^R =$ postorder(mirror(T_i)) holds for T_i

Solution:

Let $P(T)$ be "[preorder$(T)]^R =$ postorder(mirror(T))". We show $P(T)$ holds for all **CharTree**s T by structural induction.

Base case ($T = \texttt{Null}$): preorder$(T)^R = \varepsilon^R = \varepsilon =$ postorder($\texttt{Null}$) $=$ postorder(mirror($\texttt{Null}$)), so $P(\texttt{Null})$ holds.

Inductive hypothesis: Suppose $P(L) \wedge P(R)$ for arbitrary **CharTree**s L, R.

Inductive step:

We want to show $P(\texttt{CharTree}(L, c, R))$,
i.e. [preorder($\texttt{CharTree}(L, c, R)$)$]^R =$ postorder(mirror($\texttt{CharTree}(L, c, R)$)).

Let c be an arbitrary element in Σ, and let $T = \texttt{CharTree}(L, c, R)$

$$
\begin{aligned}
(T)^R &= [c \cdot (L) \cdot (R)]^R && \text{defn of preorder} \\
&= (R)^R \cdot (L)^R \cdot c^R && \text{Fact 1} \\
&= (R)^R \cdot (L)^R \cdot c && \text{Fact 2} \\
&= ((R)) \cdot ((L)) \cdot c && \text{by I.H.} \\
&= (\text{CharTree}((R), c, (L)) && \text{recursive defn of postorder} \\
&= ((\text{CharTree}(L, c, R))) && \text{recursive defn of mirror} \\
&= ((T)) && \text{defn of } T
\end{aligned}
$$

So $P(\texttt{CharTree}(L, c, R))$ holds.
By the principle of induction, $P(T)$ holds for all **CharTree**s T.