CSE 390Z: Mathematics for Computation Workshop

Week 5 Workshop

Conceptual Review

(a) **Set Definitions**

Set Equality: $A = B := \forall x(x \in A \leftrightarrow x \in B)$
Subset: $A \subseteq B := \forall x(x \in A \rightarrow x \in B)$
Union: $A \cup B := \{x \ : \ x \in A \vee x \in B\}$
Intersection: $A \cap B := \{x \ : \ x \in A \wedge x \in B\}$
Set Difference: $A \setminus B = A - B := \{x \ : \ x \in A \wedge x \notin B\}$
Set Complement: $\overline{A} = A^C := \{x \ : \ x \notin A\}$
Powerset: $\mathcal{P}(A) := \{B \ : B \subseteq A\}$
Cartesian Product: $A \times B := \{(a, b) \ : a \in A, \ b \in B\}$

(b) How do we prove that for sets A and B, $A \subseteq B$?

Solution:

Let $x \in A$ be arbitrary... thus $x \in B$. Since x was arbitrary, $A \subseteq B$.

(c) How do we prove that for sets A and B, $A = B$?

Solution:

Use two subset proofs to show that $A \subseteq B$ and $B \subseteq A$.

1. Modular Multiplication

Write an English proof to prove that for an integer $m > 0$ and any integers a, b, c, d, if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $ac \equiv bd \pmod{m}$.

Solution:

Let $m > 0$, a, b, c, d be arbitrary integers. Assume that $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$. Then by definition of mod, $m \mid (a - b)$ and $m \mid (c - d)$. Then by definition of divides, there exists some integer k such that $a - b = mk$, and there exists some integer j such that $c - d = mj$. Then $a = b + mk$ and $c = d + mj$. So, multiplying, $ac = (b + mk)(d + mj) = bd + mkd + mjb + m^2jk = bd + m(kd + jb + mjk)$. Subtracting bd from both sides, $ac - bd = m(kd + jb + mjk)$. By definition of divides, $m \mid ac - bd$. Then by definition of congruence, $ac \equiv bd \pmod{m}$.

2. Set Operations

Let $A = \{1, 2, 5, 6, 8\}$ and $B = \{2, 3, 5\}$.

(a) What is the set $A \cap (B \cup \{2, 8\})$?

Solution:

$\{2, 5, 8\}$

(b) What is the set $\{10\} \cup (A \setminus B)$?

Solution:

$\{1, 6, 8, 10\}$

(c) What is the set $\mathcal{P}(B)$?

Solution:

$\{\{2, 3, 5\}, \{2, 3\}, \{2, 5\}, \{3, 5\}, \{2\}, \{3\}, \{5\}, \emptyset\}$

(d) How many elements are in the set $A \times B$? List 3 of the elements.

Solution:

15 elements, for example $(1, 2), (1, 3), (1, 5)$.

3. A Basic Subset Proof

Prove that $A \cap B \subseteq A \cup B$.

Solution:

Let $x \in A \cap B$ be arbitrary. Then by definition of intersection, $x \in A$ and $x \in B$. So certainly $x \in A$ or $x \in B$. Then by definition of union, $x \in A \cup B$.

4. Set Equality Proof

(a) Write an English proof to show that $A \cap (A \cup B) \subseteq A$ for any sets A, B.

Solution:

Let x be an arbitrary member of $A \cap (A \cup B)$. Then by definition of intersection, $x \in A$ and $x \in A \cup B$. So certainly, $x \in A$. Since x was arbitrary, $A \cap (A \cup B) \subseteq A$.

(b) Write an English proof to show that $A \subseteq A \cap (A \cup B)$ for any sets A, B.

Solution:

Let $y \in A$ be arbitrary. So certainly $y \in A$ or $y \in B$. Then by definition of union, $y \in A \cup B$. Since $y \in A$ and $y \in A \cup B$, by definition of intersection, $y \in A \cap (A \cup B)$. Since y was arbitrary, $A \subseteq A \cap (A \cup B)$.

(c) Combine part (a) and (b) to conclude that $A \cap (A \cup B) = A$ for any sets A, B.

Solution:

Since $A \cap (A \cup B) \subseteq A$ and $A \subseteq A \cap (A \cup B)$, we can deduce that $A \cap (A \cup B) = A$.

5. Subsets

Prove or disprove: for any sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Solution:

Let A, B, C be sets, and suppose $A \subseteq B$ and $B \subseteq C$. Let x be an arbitrary element of A. Then, by definition of subset, $x \in B$, and by definition of subset again, $x \in C$. Since x was an arbitrary element of A, we see that all elements of A are in C, so by definition of subset, $A \subseteq C$. So, for any sets A, B, C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

6. $\cup \to \cap$?

Prove or disprove: for all sets A and B, $A \cup B \subseteq A \cap B$.

Solution:

We wish to disprove this claim via a counterexample. Choose $A = \{1\}$, $B = \varnothing$. Note that $A \cup B = \{1\} \cup \varnothing = \{1\}$ by definition of set union. Note that $A \cap B = \{1\} \cap \varnothing = \varnothing$ by definition of set intersection. $\{1\} \not\subseteq \varnothing$, so the claim does not hold for these sets. Since we found a counterexample to the claim, we have shown that it is not the case that $A \cup B \not\subseteq A \cap B$ for all sets A and B.

7. Set Equality Proof

Write an English proof to show that $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Solution:

Let $x \in A \setminus (B \cap C)$ be arbitrary. Then by definition of set difference, $x \in A$ and $x \notin B \cap C$. Then by definition of intersection, $x \notin B$ or $x \notin C$. Thus (by distributive property of propositions) we have $x \in A$ and $x \notin B$, or $x \in A$ and $x \notin C$. Then by definition of set difference, $x \in (A \setminus B)$ or $x \in (A \setminus C)$. Then by definition of union, $x \in (A \setminus B) \cup (A \setminus C)$. Since x was arbitrary, we have shown $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$.

Let $x \in (A \setminus B) \cup (A \setminus C)$ be arbitrary. Then by definition of union, $x \in (A \setminus B)$ or $x \in (A \setminus C)$. Then by definition of set difference, $x \in A$ and $x \notin B$, or $x \in A$ and $x \notin C$. Then (by distributive property of propositions) $x \in A$, and $x \notin B$ or $x \notin C$. Then by definition of intersection, $x \in A$ and $x \notin (B \cap C)$. Then by definition of set difference, $x \in A \setminus (B \cap C)$. Since x was arbitrary, we have shown that $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$.

Since $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$ and $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$, we have shown $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

8. Induction: A Sneak Preview

Prove that $9 \mid (n^3 + (n+1)^3 + (n+2)^3)$ for all $n > 1$ by induction.

Solution:

Let $P(n)$ be "$9 \mid n^3 + (n+1)^3 + (n+2)^3$". We will prove $P(n)$ for all integers $n > 1$ by induction.

Base Case $(n = 2)$**:** $2^3 + (2+1)^3 + (2+2)^3 = 8 + 27 + 64 = 99 = 9 \cdot 11$, so $9 \mid 2^3 + (2+1)^3 + (2+2)^3$, so $P(2)$ holds.

Inductive Hypothesis: Assume that $9 \mid k^3 + (k+1)^3 + (k+2)^3$ for an arbitrary integer $k > 1$. Note that this is equivalent to assuming that $k^3 + (k+1)^3 + (k+2)^3 = 9j$ for some integer j by the definition of divides.

Inductive Step: Goal: Show $9 \mid (k+1)^3 + (k+2)^3 + (k+3)^3$

$$
\begin{aligned}
(k+1)^3 + (k+2)^3 + (k+3)^3 &= (k^2 + 6k + 9)(k+3) + (k+1)^3 + (k+2)^3 && \text{[expanding trinomial]} \\
&= (k^3 + 6k^2 + 9k + 3k^2 + 18k + 27) + (k+1)^3 + (k+2)^3 && \text{[expanding binomial]} \\
&= 9k^2 + 27k + 27 + k^3 + (k+1)^3 + (k+2)^3 && \text{[adding like terms]} \\
&= 9k^2 + 27k + 27 + 9j && \text{[by I.H.]} \\
&= 9(k^2 + 3k + 3 + j) && \text{[factoring out 9]}
\end{aligned}
$$

Since k and j are integers, $k^2 + 3k + 3 + j$ is also an integer. Therefore, by the definition of divides, $9 \mid (k+1)^3 + (k+2)^3 + (k+3)^3$, so $P(k) \to P(k+1)$ for an arbitrary integer $k > 1$.

Conclusion: $P(n)$ holds for all integers $n > 1$ by induction.