CSE 390Z: Mathematics of Computing Workshop

Week 8 Workshop

0. Conceptual Review

(a) Regular expression rules:
Basis: ϵ, a for $a \in \Sigma$
Recursive: If A, B are regular expressions, $(A \cup B), AB$, and A^* are regular expressions.

1. Structural Induction: Divisible by 4

Define a set $\mathfrak{B}$ of numbers by:

- 4 and 12 are in $\mathfrak{B}$
- If $x \in \mathfrak{B}$ and $y \in \mathfrak{B}$, then $x + y \in \mathfrak{B}$ and $x - y \in \mathfrak{B}$

Prove by induction that every number in $\mathfrak{B}$ is divisible by 4.

Complete the proof below:

2. Structural Induction: CharTrees

Recursive Definition of CharTrees:

- Basis Step: `Null` is a **CharTree**
- Recursive Step: If L, R are **CharTrees** and $c \in \Sigma$, then $\texttt{CharTree}(L, c, R)$ is also a **CharTree**

Intuitively, a **CharTree** is a tree where the non-null nodes store a `char` data element.

Recursive functions on CharTrees:

- The preorder function returns the preorder traversal of all elements in a **CharTree**.

$$\begin{aligned} \text{preorder}(\texttt{Null}) &= \varepsilon \\ \text{preorder}(\texttt{CharTree}(L, c, R)) &= c \cdot \text{preorder}(L) \cdot \text{preorder}(R) \end{aligned}$$

- The postorder function returns the postorder traversal of all elements in a **CharTree**.

$$\begin{aligned} \text{postorder}(\texttt{Null}) &= \varepsilon \\ \text{postorder}(\texttt{CharTree}(L, c, R)) &= \text{postorder}(L) \cdot \text{postorder}(R) \cdot c \end{aligned}$$

- The mirror function produces the mirror image of a **CharTree**.

$$\begin{aligned} \text{mirror}(\texttt{Null}) &= \texttt{Null} \\ \text{mirror}(\texttt{CharTree}(L, c, R)) &= \texttt{CharTree}(\text{mirror}(R), c, \text{mirror}(L)) \end{aligned}$$

- Finally, for all strings x, let the "reversal" of x (in symbols x^R) produce the string in reverse order.

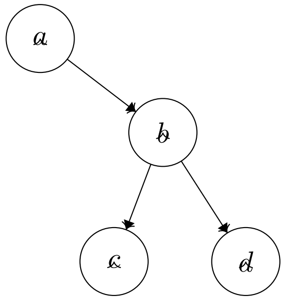
Additional Facts:
You may use the following facts:

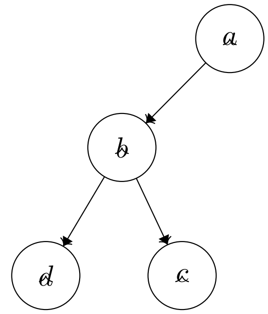
- For any strings $x_1, ..., x_k$: $(x_1 \cdot ... \cdot x_k)^R = x_k^R \cdot ... \cdot x_1^R$
- For any character c, $c^R = c$

Statement to Prove:
Show that for every **CharTree** T, the reversal of the preorder traversal of T is the same as the postorder traversal of the mirror of T. In notation, you should prove that for every **CharTree**, T: $[\text{preorder}(T)]^R = \text{postorder}(\text{mirror}(T))$.

There is an example and space to work on the next page.

Example for Intuition:





Let T_i be the tree above.
preorder(T_i) =“abcd”.
T_i is built as $(\texttt{null}, a, U)$
Where U is (V, b, W),
$V = (\texttt{null}, c, \texttt{null}), W = (\texttt{null}, d, \texttt{null})$.

This tree is mirror(T_i).
postorder(mirror(T_i)) =“dcba”,
“dcba” is the reversal of “abcd” so
[preorder(T_i)]^R = postorder(mirror(T_i)) holds for T_i

3. Regular Expressions Warmup

Consider the following Regular Expression (RegEx):

$$1(45 \cup 54)^{\star}1$$

List 5 strings accepted by the RegEx and 5 strings from $T := \{1, 4, 5\}^{\star}$ rejected by the RegEx. Then, summarize this RegEx in your own words.

4. Context Free Grammars Warmup

Consider the following CFG which generates strings from the language V $:= \{0, 1, 2, 3, 4\}^{*}$

$$\mathbf{S} \rightarrow 0\mathbf{X}4$$
$$\mathbf{X} \rightarrow 1\mathbf{X}3 \mid 2$$

List 5 strings generated by the CFG and 5 strings from V not generated by the CFG. Then, summarize this CFG in your own words.

5. Constructing RegExs and CFGs

For each of the following, construct a regular expression and CFG for the specified language.

(a) Strings from the language $S := \{a\}^*$ with an even number of a's.

(b) Strings from the language $S := \{a, b\}^*$ with an even number of a's.

(c) Strings from the language $S := \{a, b\}^*$ with odd length.

(d) (Challenge) Strings from the language $S := \{a, b\}^*$ with an even number of a's or an odd number of b's.

6. Structural Induction: CFGs

Consider the following CFG:

$$S \rightarrow SS \mid 0S1 \mid 1S0 \mid \epsilon$$

Prove that every string generated by this CFG has an equal number of 1's and 0's.

Hint 1: Start by converting this CFG to a recursively defined set.
Hint 2: You may wish to define the functions $\#_0(x), \#_1(x)$ on a string x.

7. Bijections

Write a proof to show that both of these functions are a bijection from $\mathbb{R}$ to $\mathbb{R}$.

(a) $f(x) = 2x + 1$

(b) $f(x) = x^3$