CSE 390Z: Mathematics for Computation Workshop

Week 3 Workshop Problems

Conceptual Review

(a) Inference Rules:

Introduce $\vee$:	$\dfrac{A}{\therefore\ A\vee B,\ B\vee A}$	Eliminate $\vee$:	$\dfrac{A\vee B\ ;\ \neg A}{\therefore\ B}$
Introduce $\wedge$:	$\dfrac{A\ ;\ B}{\therefore\ A\wedge B}$	Eliminate $\wedge$:	$\dfrac{A\wedge B}{\therefore\ A\ ,\ B}$
Direct Proof:	$\dfrac{A\Rightarrow B}{\therefore\ A\rightarrow B}$	Modus Ponens:	$\dfrac{A\ ;\ A\rightarrow B}{\therefore\ B}$
Intro $\exists$:	$\dfrac{P(c)\text{ for some }c}{\therefore\ \exists x P(x)}$	Eliminate $\exists$:	$\dfrac{\exists x P(x)}{\therefore\ P(c)\text{ for a fresh }c}$
Intro $\forall$:	$\dfrac{P(a);\ a\text{ is arbitrary}}{\therefore\ \forall x P(x)}$	Eliminate $\forall$:	$\dfrac{\forall x P(x)}{\therefore\ P(a);\ a\text{ is arbitrary}}$

(b) Given $A \wedge B$, prove $A \vee B$

Given $P \rightarrow R$, $R \rightarrow S$, prove $P \rightarrow S$.

(c) How do you prove a "for all" statement in an English proof? E.g. prove $\forall x P(x)$. How do you prove a "there exists" statement? E.g. prove $\exists x P(x)$.

(d) What's the definition of "a divides b"?

1. Translations with Integers

Translate the following English sentences to predicate logic. The domain is integers, and you may use $=$, $\neq$, and $>$ as predicates. Assume the predicates Prime, Composite, and Even have been defined appropriately.
Note: Composite numbers are ones that have at least 2 factors (the opposite of prime).

(a) 2 is prime.

(b) Every **positive** integer is prime or composite, but not both.

(c) There is **exactly one** even prime.

(d) 2 is the only even prime.

(e) Some, but not all, composite integers are even.

2. Formal Proofs: Modus Ponens

(a) Prove that given $p \rightarrow q$, $\neg s \rightarrow \neg q$, and p, we can conclude s.

(b) Prove that given $\neg(p \vee q) \rightarrow s$, $\neg p$, and $\neg s$, we can conclude q.

3. Formal Proofs: Direct Proof Rule

(a) Prove that given $p \rightarrow q$, we can conclude $(p \wedge r) \rightarrow q$

(b) Prove that given $p \vee q$, $q \rightarrow r$, and $r \rightarrow s$, we can conclude $\neg p \rightarrow s$.

4. Predicate Logic Formal Proof

(a) Prove that $\forall x P(x) \rightarrow \exists x P(x)$. You may assume that the domain is nonempty.

(b) Given $\forall x(T(x) \rightarrow M(x))$ and $\exists x(T(x))$, prove that $\exists x(M(x))$.

(c) Given $\forall x(P(x) \rightarrow Q(x))$, prove that $(\exists x P(x)) \rightarrow (\exists y Q(y))$.

5. A Rational Conclusion

Note: This problem will walk you through the steps of an English proof. If you feel comfortable writing the proof already, feel free to jump directly to part (h).

Let the predicate Rational(x) be defined as $\exists a \exists b(\text{Integer}(a) \wedge \text{Integer}(b) \wedge b \neq 0 \wedge x = \frac{a}{b})$. Prove the following claim:

$$\forall x \forall y(\text{Rational}(x) \wedge \text{Rational}(y) \wedge (y \neq 0) \rightarrow \text{Rational}(\frac{x}{y}))$$

(a) Translate the claim to English.

(b) State the givens and declare any arbitrary variables you need to use.
Hint: there are no givens in this problem.

(c) State the assumptions you're making.
Hint: assume everything on the left side of the implication.

(d) Unroll the predicate definitions from your assumptions.

(e) Manipulate what you have towards your goal (might be easier to do the next step first).

(f) Reroll into your predicate definitions.

(g) State your final claim.

(h) Now take these proof parts and assemble them into one cohesive English proof.

6. Oddly Even

(a) Write a formal proof to show: If n, m are odd, then $n + m$ is even.

Let the predicates $\mathsf{Odd}(x)$ and $\mathsf{Even}(x)$ be defined as follows where the domain of discourse is integers:

$$\mathsf{Odd}(x) := \exists y\ (x = 2y + 1)$$
$$\mathsf{Even}(x) := \exists y\ (x = 2y)$$

(b) Prove the same statement from part (a) using an English proof.

7. Divisibility Proof

Let the domain of discourse be integers. Consider the following claim:

$$\forall n \forall d\ ((d \mid n) \rightarrow (-d \mid n))$$

(a) Translate the claim into English.

(b) Write a formal proof to show that the claim holds.

(c) Translate your proof to English.

8. Challenge: Divides Proof

Write an English proof to prove that if k is an odd integer, then $4 \mid k^2 - 1$.

9. Challenge: Formal Proof

Given $\forall x\ (P(x) \vee Q(x))$ and $\forall y\ (\neg Q(y) \vee R(y))$, prove $\exists x\ (P(x) \vee R(x))$. You may assume that the domain is not empty.