

## CSE 390Z: Mathematics for Computation Workshop

### QuickCheck: Induction Solutions (due Monday, November 11)

Please submit a response to the following questions on Gradescope. We do not grade on accuracy, so please submit your best attempt. You may either typeset your responses or hand-write them. Note that hand-written solutions must be legible to be graded.

We have created [this template](#) if you choose to typeset with Latex. [This guide](#) has specific information about scanning and uploading pdf files to Gradescope.

#### 0. Induction: Equality

For any  $n \in \mathbb{N}$ , define  $S_n$  to be the sum of the squares of the first  $n$  positive integers, or

$$S_n = 1^2 + 2^2 + \dots + n^2.$$

Prove that for all  $n \in \mathbb{N}$ ,  $S_n = \frac{1}{6}n(n+1)(2n+1)$ .

##### Solution:

Let  $P(n)$  be the statement " $S_n = \frac{1}{6}n(n+1)(2n+1)$ " defined for all  $n \in \mathbb{N}$ . We prove that  $P(n)$  is true for all  $n \in \mathbb{N}$  by induction on  $n$ .

**Base Case:** When  $n = 0$ , we know the sum of the squares of the first  $n$  positive integers is the sum of no terms, so we have a sum of 0. Thus,  $S_0 = 0$ . Since  $\frac{1}{6}(0)(0+1)((2)(0)+1) = 0$ , we know that  $P(0)$  is true.

**Inductive Hypothesis:** Suppose that  $P(k)$  is true for some arbitrary  $k \in \mathbb{N}$ .

##### Inductive Step:

Goal: Show  $P(k+1)$ , i.e. show  $S_{k+1} = \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$

Examining  $S_{k+1}$ , we see that

$$S_{k+1} = 1^2 + 2^2 + \dots + k^2 + (k+1)^2 = S_k + (k+1)^2.$$

By the inductive hypothesis, we know that  $S_k = \frac{1}{6}k(k+1)(2k+1)$ . Therefore, we can substitute and rewrite the expression as follows:

$$\begin{aligned} S_{k+1} &= S_k + (k+1)^2 \\ &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\ &= (k+1) \left( \frac{1}{6}k(2k+1) + (k+1) \right) \\ &= \frac{1}{6}(k+1)(k(2k+1) + 6(k+1)) \\ &= \frac{1}{6}(k+1)(2k^2 + 7k + 6) \\ &= \frac{1}{6}(k+1)(k+2)(2k+3) \\ &= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1) \end{aligned}$$

Thus, we can conclude that  $P(k+1)$  is true.

**Conclusion:**  $P(n)$  holds for all integers  $n \geq 0$  by the principle of induction.

## 1. Video Solution

Watch [this video](#) on the solution **after** making an initial attempt. Then, answer the following questions.

- (a) What is one thing you took away from the video solution?