CSE 390Z: Mathematics for Computation Workshop

Practice 311 Final Solutions

Name: ______________________________

UW ID: ______________________________

Instructions:

- This is a **simulated practice final**. You will **not** be graded on your performance on this exam.
- Nevertheless, please treat this as if it is a real exam. That means that you may not discuss with your neighbors, reference outside material, or use your devices during the next 60 minute period.
- If you get stuck on a problem, consider moving on and coming back later. In the actual exam, there will likely be opportunity for partial credit.
- There are 5 problems on this exam, totaling 100 points.

1. True or False [30 points]

For the following questions, determine whether the statement is true or false. Then provide 1-3 sentences of explanation. Your explanations **do not** need to be full or formal proofs.

(a) (5 points) "p only if q" and "q is necessary for p", are both best translated as $p \to q$.

Solution:

True. Both translations say that in order for p to occur, q must occur. So when p happens q must have happened too. Thus $p \to q$.

(b) (5 points) One way to prove that $p \to q$ is true is to show that the converse, $q \to p$, is false.

Solution:

False. There are implications where both the converse and the original statement are true, so this is not a valid proof technique.

(c) (5 points) The implication $\forall y \exists x$ P$(x, y) \to \exists x \forall y$ P(x, y) is true regardless of what the predicate P is.

Solution:

False. In the statement $\forall y \exists x$ P(x, y), it may be that x depends on the value of y. For example, "every person has an ancestor". In the statement $\exists x \forall y$ P(x, y), it must be that the single x satisfy P for all y. For example, "there is one person that is every person's ancestor." So, the first statement does not always imply the second.

(d) (5 points) Suppose a, b, m, n are all integers greater than 1. If $a \equiv_m b$ and $m \mid n$, then $a \equiv_n b$.

Solution:

False. For example, consider $a = 3$, $b = 6$, $m = 3$, and $n = 9$. Then $a \equiv_m b$ and $m \mid n$ but $a \not\equiv_n b$

(e) (5 points) Suppose A, B are sets. If $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A \subseteq B$.

Solution:

True. Let $x \in A$ be arbitrary. Consider $\{x\}$, i.e. the singleton set containing just the element x. Since $x \in A$, by definition of subset $\{x\} \subseteq A$. Then by definition of powerset, $\{x\} \in \mathcal{P}(A)$. Then because $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, we have $\{x\} \in \mathcal{P}(B)$. Then by definition of powerset, $\{x\} \subseteq B$. Then by definition of subset, every element of $\{x\}$ is an element of B. In particular, $x \in B$. Since x was arbitrary, $A \subseteq B$.

Note: This level of detail and formalism is not required.

(f) (5 points) Strong induction proofs always require more than one base case.

Solution:

False. Not necessarily. For example, the first strong induction proof we did in class (every positive integer greater than 1 can be written as a product of primes) only required one base case.

2. All the Machines! [15 points]
Let the alphabet be $\Sigma = \{a, b\}$. Consider the language $L = \{w \in \Sigma^* : \text{every } a \text{ has a } b \text{ two characters later}\}$. In other words, L is the language of all strings in the alphabet a, b where after any a, the character after the a can be anything, but the character after that one must be a b.

Some strings in L include ε, abb, $aabb$, $bbbbabb$. Some strings not in L include a, ab, aab, $ababb$. Notice that the last two characters of the string cannot be an a.

(g) (5 points) Give a regular expression that represents L.

Solution:

$(b \cup abb \cup aabb)^*$

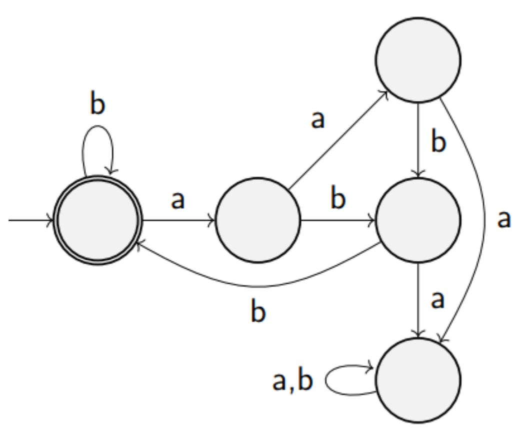
(h) (5 points) Give a CFG that represents L.

Solution:

S$\rightarrow$ b**S** | $aabb$**S** | abb**S** | ε

(i) (5 points) Give a DFA that represents L.

Solution:



3. Induction I (20 points)

Let the function $f : \mathbb{N} \to \mathbb{N}$ be defined as follows:

$f(0) = 2$
$f(1) = 7$
$f(n) = f(n-1) + 2f(n-2)$ for $n \geq 2$
Prove that $f(n) = 3 * 2^n + (-1)^{n+1}$ for all integers $n \geq 0$ using strong induction.

Solution:
Let $P(n)$ be "$f(n) = 3 * 2^n + (-1)^{n+1}$. We will prove $P(n)$ holds for all $n \geq 0$ by strong induction.

Base Cases:
$n = 0$:
$f(0) = 2$
$3 * 2^0 + (-1)^{0+1} = 3 * 1 + (-1)^1 = 3 + (-1) = 2.$
$2 = 2$, so $P(0)$ holds.

$n = 1$:
$f(1) = 7$
$3 * 2^1 + (-1)^{1+1} = 3 * 2 + (-1)^2 = 6 + 1 = 7$
$7 = 7$, so $P(1)$ holds.

Inductive Hypothesis: Suppose that $P(j)$ holds for all $0 \leq j \leq k$ for some arbitrary integer k.
Inductive Step:

Goal: Show $P(k+1)$, i.e. show $f(k+1) = 3 * 2^{k+1} + (-1)^{(k+1)+1}$

$$
\begin{aligned}
f(k+1) &= f((k+1)-1) + 2f((k+1)-2) && \text{def. of } f \\
&= f(k) + 2f(k-1) \\
&= 3 * 2^k + (-1)^{k+1} + 2(3 * 2^{k-1} + (-1)^{(k-1)+1}) && \text{IH} \\
&= 3 * 2^k + (-1)^{k+1} + 2(3 * 2^{k-1} + (-1)^k) \\
&= 3 * 2^k + (-1)^{k+1} + 3 * 2^k + 2(-1)^k \\
&= 2 * 3 * 2^k + (-1)^{k+1} + 2(-1)^k \\
&= 3 * 2^{k+1} + (-1)^{k+1} + 2(-1)^k \\
&= 3 * 2^{k+1} + (-1)(-1)(-1)^{k+1} + 2(-1)(-1)(-1)^k && (-1)(-1) = 1 \\
&= 3 * 2^{k+1} - (-1)^{k+2} + 2(-1)^{k+2} \\
&= 3 * 2^{k+1} + (-1)^{(k+1)+1}
\end{aligned}
$$

Thus, $P(k+1)$ holds.

We conclude that $P(n)$ holds for all $n \geq 0$ by strong induction.

4. Induction II (20 points)

Let the set S be recursively defined as follows:
Basis: $(0,0) \in S$
Recursive Step: If $(x,y) \in S$, then $(x+2, y+4) \in S$ and $(x+4, y+8) \in S$.

Prove that for all $(x,y) \in S$, $x+y$ is divisible by 3.

Solution:
Define $P((x,y))$ to be the claim $3 \mid (x+y)$. We will prove that $P((x,y))$ holds for all $(x,y) \in S$ by structural induction.
Base Case: $(x,y) = (0,0)$
$0+0 = 0 = 3*0$. So, $3|(x+y)$ and $P((0,0))$ holds.

Inductive Hypothesis: Suppose that $P((a,b))$ holds for some arbitrary $(a,b) \in S$. (i.e. $3|(a+b)$).

Inductive Step:

Goal: Show $P((a+2, b+4))$ and $P((a+4, b+8))$

By the inductive hypothesis, $3 \mid (a+b)$. By definition of divides, $3k = a+b$ for some integer k.

$$(a+2)+(b+4) = a+b+6 = 3k+6 = 3(k+2)$$

So, $3 \mid ((a+2)+(b+4))$ which means $P((a+2, b+4))$ holds.

$$(a+4)+(b+8) = a+b+12 = 3k+12 = 3(k+4)$$

So, $3 \mid ((a+4)+(b+8))$ which means $P((a+4, b+8))$ holds.

Thus, $P((x,y))$ holds for all $(x,y) \in S$ by the principle of structural induction.

5. Irregularity (15 points)
Prove that the language $L = \{10^x10^{x+1}1 : x \geq 0\}$ is not regular.

Solution:
Suppose for the sake of contradiction there exists a DFA D that accepts L.

Let $S = \{10^n1 : n \geq 0\}$. Since S contains infinitely number strings and D has a finite number of states, two strings in S must end up in the same state of D. Say those strings are 10^i1 and 10^j1 for some $i, j \geq 0$ where $i \neq j$. Now, append $0^{i+1}1$ to both strings. The resulting strings are:

$x = 10^i10^{i+1}1$. Note that $x \in L$.
$y = 10^j10^{i+1}1$. Note that $y \notin L$ since $i \neq j, j+1 \neq i+1$.

Both x and y must end up at the same state, but since $x \in L$ and $y \notin L$, that state must be both an accept state and a reject state. This is a contradiction, which means there does not exist a DFA D which accepts L. This means L is not regular.