# CSE 390Z: Mathematics for Computation Workshop

## Mid-Quarter Review Solutions

Name: ______________________________

## 0. Training Wheels

For this problem, our domain of discourse is college football teams and college football conferences.
You are allowed to use the $\neq$ symbol to check that two objects are not equivalent.
We will use the following predicates:

- $\texttt{Team}(x) := x$ is a football team.
- $\texttt{UW}(x) := x$ is the University of Washington football team.
- $\texttt{WSU}(x) := x$ is the Washington State University football team.
- $\texttt{OSU}(x) := x$ is the Oregon State University football team.
- $\texttt{OldPac}(x) := x$ is the old Pac-12 Conference.
- $\texttt{NewPac}(x) := x$ is the new Pac-2 Conference.
- $\texttt{Member}(x, y) :=$ the football team $x$ has been a part of the conference $y$.
- $\texttt{Lost}(x, y) := x$ lost to $y$ in a football game.

(a) State whether the two statements below are equivalent. Provide a one sentence justification.

$$\exists y\Big[\texttt{OldPac}(y) \land \forall x\Big(\texttt{Team}(x) \to \big(\texttt{UW}(x) \to \texttt{Member}(x, y)\big)\Big)\Big]$$
$$\exists y\Big[\texttt{OldPac}(y) \land \forall x\Big(\texttt{UW}(x) \to \texttt{Member}(x, y)\Big)\Big]$$

**Solution:**

Yes, they are equivalent. The UW Huskies are already a football team, so there is no need to actually state $\texttt{Team}(x)$.

(b) Translate the following sentence into predicate logic.

Excluding WSU, at least one team has been a part of the new Pac-2 conference and the old Pac-12 conference.

**Solution:**

$$\exists x \exists y \exists z\Big[\texttt{Team}(x) \land \neg\texttt{WSU}(x) \land \texttt{OldPac}(y) \land \texttt{Member}(x, y) \land \texttt{NewPac}(z) \land \texttt{Member}(x, z)\Big]$$

(c) Translate the following statement into predicate logic.

UW has won against all football teams besides itself, and WSU has lost to all foootball teams besides itself.

**Solution:**

$$\exists x \exists y \Big[ \texttt{UW}(x) \land \forall a\big( (\texttt{Team}(a) \land (a \neq x)) \to \texttt{Lost}(a,x)\big) \land$$
$$\texttt{WSU}(y) \land \forall b\big( (\texttt{Team}(b) \land (b \neq y)) \to \texttt{Lost}(y,b)\big)\Big]$$

(d) Negate the following statement. Your final answer should have zero negations.

Warning: this statement makes absolutely no sense. Do **NOT** spend time thinking about its meaning. We want you to blindly follow your equivalency laws here.

$$\forall x \forall y \Big[\big(\texttt{WSU}(x) \land \texttt{OSU}(y)\big) \land \big(\neg\texttt{Lost}(x,y) \lor \neg\texttt{Lost}(y,x)\big)\Big]$$

**Solution:**

Students only need to write the final answer to receive full credit.

$$\exists x \exists y \Big[\big(\texttt{WSU}(x) \land \texttt{OSU}(y)\big) \to \big(\texttt{Lost}(x,y) \land \texttt{Lost}(y,x)\big)\Big]$$

The corresponding chain of equivalences is provided below for reference.

$$\neg\forall x \forall y \Big[\big(\texttt{WSU}(x) \land \texttt{OSU}(y)\big) \land \big(\neg\texttt{Lost}(x,y) \lor \neg\texttt{Lost}(y,x)\big)\Big]$$

$$\equiv \neg\forall x \forall y \Big[\big(\texttt{WSU}(x) \land \texttt{OSU}(y)\big) \land \neg\big(\texttt{Lost}(x,y) \land \texttt{Lost}(y,x)\big)\Big] \qquad \text{DeMorgans}$$

$$\equiv \exists x \exists y \neg\Big[\big(\texttt{WSU}(x) \land \texttt{OSU}(y)\big) \land \neg\big(\texttt{Lost}(x,y) \land \texttt{Lost}(y,x)\big)\Big] \qquad \text{DeMorgans for Quantifiers}$$

$$\equiv \exists x \exists y \Big[\neg\big(\texttt{WSU}(x) \land \texttt{OSU}(y)\big) \lor \neg\neg\big(\texttt{Lost}(x,y) \land \texttt{Lost}(y,x)\big)\Big] \qquad \text{DeMorgans}$$

$$\equiv \exists x \exists y \Big[\neg\big(\texttt{WSU}(x) \land \texttt{OSU}(y)\big) \lor \big(\texttt{Lost}(x,y) \land \texttt{Lost}(y,x)\big)\Big] \qquad \text{Double Negation}$$

$$\equiv \exists x \exists y \Big[\big(\texttt{WSU}(x) \land \texttt{OSU}(y)\big) \to \big(\texttt{Lost}(x,y) \land \texttt{Lost}(y,x)\big)\Big] \qquad \text{Law of Implication}$$

## 1. Good Ol' Proofs

(a) Prove for some predefined sets $A, B, C$ that $(A \setminus B) \cup (C \setminus B) = (A \cup C) \setminus B$.

*Hint*: You will need to use proof by cases.

**Solution:**

($\subseteq$)

Let $x \in (A \setminus B) \cup (C \setminus B)$ be arbitrary. We will do a proof by cases on whether $x \in A \setminus B$ or $x \in C \setminus B$.

Case 1: Suppose $x \in A \setminus B$. By definition, $x \in A$ and $x \notin B$. Since $x \in A$, surely $x \in A \cup C$. Since $x \in A \cup C$ and $x \notin B$, we have $x \in (A \cup C) \setminus B$.

Case 2: Suppose $x \in C \setminus B$. By definition, $x \in C$ and $x \notin B$. Since $x \in C$, surely $x \in C \cup A$, or equivalently $x \in A \cup C$. Since $x \in A \cup C$ and $x \notin B$, we have $x \in (A \cup C) \setminus B$.

Since $x$ was an arbitrary, we have shown that $(A \setminus B) \cup (C \setminus B) \subseteq (A \cup C) \setminus B$.

($\supseteq$)

Let $x \in (A \cup C) \setminus B$ be arbitary. By definition, $x \in A \cup C$ and $x \notin B$. We will do a proof by cases on whether $x \in A$ or $x \in C$.

Case 1: Suppose $x \in A$. Since $x \in A$ and $x \notin B$, we have $x \in A \setminus B$. Then, we can also say $x \in (A \setminus B) \cup (C \setminus B)$.

Case 2: Suppose $x \in C$. Since $x \in C$ and $x \notin B$, we have $x \in C \setminus B$. Then, we can also say $x \in (C \setminus B) \cup (A \setminus B)$, or equivalently $x \in (A \setminus B) \cup (C \setminus B)$.

Since $x$ was an arbitrary, we have shown that $(A \cup C) \setminus B \subseteq (A \setminus B) \cup (C \setminus B)$.

Therefore, since we have proven that $(A \setminus B) \cup (C \setminus B) \subseteq (A \cup C) \setminus B$ and $(A \cup C) \setminus B \subseteq (A \setminus B) \cup (C \setminus B)$, we have shown that $(A \setminus B) \cup (C \setminus B) = (A \cup C) \setminus B$.

(b) Prove true or false: if $a \equiv 1 \pmod 5$ and $b \equiv 1 \pmod 5$, then $\gcd(a, b) \equiv 1 \pmod 5$.

**Solution:**

False. For example, when $a = 6$ and $b = 21$, their $\gcd$ is 3 and clearly $3 \not\equiv 1 \pmod 5$.

(c) Consider the following statement: if $a, b \in \mathbb{Z}$ and $a \geq 2$, then $a \nmid b$ or $a \nmid b + 1$.

Your goal is to disprove this statement using proof by contradiction.

Write out **JUST THE FIRST SENTENCE** of the proof.

Your answer should look something like "Suppose, for the sake of contradiction, ..."

**Solution:**

Suppose, for the sake of contradiction, that there exist $a, b \in \mathbb{Z}$ with $a \geq 2$ where $a \mid b$ and $a \mid b + 1$.

## 2. Modular Arithmetic

Prove that for all integers $x, y, n > 0$, if $x \equiv_{6n} 1$ and $y \equiv_{7n} 5$ then $7x + 2y \equiv_{14n} 17$.

Hint: Apply the definition of congruence and divides.

**Solution:**

Let $x, y, n > 0$ be arbitrary integers. Suppose $x \equiv_{6n} 1$ and $y \equiv_{7n} 5$. Then by definition of congruence, $6n \mid (x - 1)$ and $7n \mid (y - 5)$. Then by definition of divides, there exists integers $j, k$ such that $x - 1 = 6nk$ and $y - 5 = 7nj$. Thus $x = 6nk + 1$ and $y = 7nj + 5$. Then observe:

$$\begin{aligned} 7x + 2y &= 7(6nk + 1) + 2(7nj + 5) \\ &= 42nk + 7 + 14nj + 10 \\ &= 42nk + 14nj + 17 \\ &= 14n(3k + j) + 17 \end{aligned}$$

Then $(7x + 2y) - 17 = 14n(3k + j)$. Since $k, j$ are integers, $3k + j$ is an integer. So by definition of divides, $14n \mid (7x + 2y) - 17$. Then by definition of congruence, $7x + 2y \equiv_{14n} 17$. Since $x, y, n$ were arbitrary, the claim holds.

## 3. Induction

Prove by induction that $3^n - 1$ is divisible by 2 for any integer $n \geq 1$.

**Solution:**

1. Let P($n$) be the statement "$3^n - 1$ is divisible by $2$". We prove P($n$) for all integers $n \geq 1$ by induction.

2. Base Case: When $n = 1$, $3^n - 1 = 3^1 - 1 = 3 - 1 = 2$. Since $2 \mid 2$, the base case holds.

3. Inductive Hypothesis: Suppose that P($k$) holds for some arbitrary integer $k \geq 1$. Then $2 \mid 3^k - 1$. Then by definition of divides, there exists some integer $a$ such that $3^k - 1 = 2a$.

4. Inductive Step: Observe that...

$$
\begin{aligned}
3^{k+1} - 1 &= 3(3^k) - 1 && \text{Definition of Exponent} \\
&= 3(3^k - 1 + 1) - 1 && \text{Subtract and Add by 1} \\
&= 3(2a + 1) - 1 && \text{By IH} \\
&= 6a + 3 - 1 && \text{Algebra} \\
&= 6a + 2 && \text{Algebra} \\
&= 2(3a + 1) && \text{Algebra}
\end{aligned}
$$

Thus by definition of divides, $2 \mid 3^{k+1} - 1$. So P($k + 1$) holds.

5. Thus we have proven P($n$) for all integers $n \geq 1$ by induction.

## 4. Strong Induction

Consider the function $f$, which takes a natural number as input and outputs a natural number.

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ f(n-1) + 2 \cdot f(n-2) & \text{if } n \geq 2 \end{cases}$$

Prove that $f(n) = 2^n$ for all $n \in \mathbb{N}$.

**Solution:**

Let $P(n)$ be the claim that $f(n) = 2^n$. We will prove $P(n)$ true for all $n \in \mathbb{N}$ by strong induction.

- $f(0) = 1 = 1 = 2^0$ so $P(0)$ holds.
- $f(1) = 2 = 2 = 2^1$ so $P(1)$ holds.

Suppose that for some arbitrary integer $k \geq 1$, that $P(j)$ holds for all $j \in \mathbb{N}$ such that $j \leq k$.
Show $P(k+1)$, i.e. $f(k+1) = 2^{k+1}$

$$\begin{aligned} f(k+1) &= f(k+1-1) + 2 \cdot f(k+1-2) && \text{Definition of } f \\ &= f(k) + 2 \cdot f(k-1) \\ &= 2^k + 2 \cdot 2^{k-1} && \text{I.H. twice} \\ &= 2^k + 2^{k-1+1} \\ &= 2^k + 2^k \\ &= 2 \cdot 2^k \\ &= 2^{k+1} \end{aligned}$$

Clearly, $P(k+1)$ holds.
Therefore, we have proven the claim true by strong induction.