CSE 390Z: Mathematics for Computation Workshop

Practice 311 Final Solutions

Name: ______________________________

UW ID: ______________________________

Instructions:

- This is a **simulated practice final**. You will **not** be graded on your performance on this exam.
- This final was written to take 50 minutes. The real final will be 110 minutes.
- Nevertheless, please treat this as if it is a real exam. That means that you may not discuss with your neighbors, reference outside material, or use your devices during the next one hour period.
- If you get stuck on a problem, consider moving on and coming back later. In the actual exam, there will likely be opportunity for partial credit.
- There are 5 problems on this exam.

1. Predicate Translation [20 points]
Let the domain of discourse be people. Translate the following statements to predicate logic, using the following predicates:

City(x) := x is a city
Suburb(x) := x is a suburb
Town(x) := x is a town
Person(x) := x is a person
LivesIn(x, y) := x lives in y
WorksIn(x, y) := x works in y

You may also use $=$ and $\neq$ as predicates.

(a) (5 points) Alice lives in exactly one city.

Solution:
$\exists x\Big(\text{City}(x) \land \text{LivesIn}(\text{Alice}, x) \land \forall y\big[\big(\text{City}(y) \land \text{LivesIn}(\text{Alice}, y)\big) \to (y = x)\big]\Big)$
OR
$\exists x\Big(\text{City}(x) \land \text{LivesIn}(\text{Alice}, x) \land \forall y\big[(y \neq x) \to\big(\neg\text{City}(y) \lor \neg\text{LivesIn}(\text{Alice}, y)\big)\big]\Big)$

(b) (5 points) Everyone who lives in a suburb works in a city.

Solution:
$\forall x\big(\big[\text{Person}(x) \land \exists y(\text{Suburb}(y) \land \text{LivesIn}(x, y))\big] \to \exists z(\text{City}(z) \land \text{WorksIn}(x, z))\big)$

(c) (5 points) There's a town that no one lives in.

Solution:
$\exists x(\text{Town}(x) \land \neg\exists y(\text{Person}(y) \land \text{LivesIn}(y, x)))$
OR
$\exists x(\text{Town}(x) \land \forall y(\text{Person}(y) \to \neg\text{LivesIn}(y, x)))$

(d) (5 points) Every city is also a town, but not every town is a city.

Solution:
$\forall x(\text{City}(x) \to \text{Town}(x)) \land \exists x(\text{Town}(x) \land \neg\text{City}(x))$
OR
$\forall x(\text{City}(x) \to \text{Town}(x)) \land \neg\forall x(\text{Town}(x) \to \text{City}(x))$

2. All the Machines! [15 points]

Let the alphabet be $\Sigma = \{a, b\}$. Consider the language $L = \{w \in \Sigma^* :$ every a has a b two characters later$\}$. In other words, L is the language of all strings in the alphabet a, b where after any a, the character after the a can be anything, but the character after that one must be a b.

Some strings in L include ε, abb, $aabb$, $bbbbabb$. Some strings not in L include a, ab, aab, $ababb$. Notice that the last two characters of the string cannot be an a.

(a) (5 points) Give a regular expression that represents L.

Solution:

$(b \cup abb \cup aabb)^*$

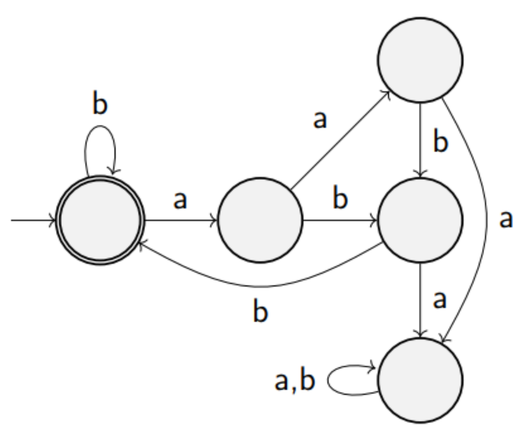
(b) (5 points) Give a CFG that represents L.

Solution:

S$\rightarrow$ b**S** | $aabb$**S** | abb**S** | ε

(c) (5 points) Give a DFA that represents L.

Solution:



3. Induction [20 points]
Consider the following recursive definition of a_n:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1 \\ a_n &= \frac{1}{2}(a_{n-1} + \frac{2}{a_{n-2}}) && \text{for } n > 2 \end{aligned}$$

Prove that $1 \leq a_n \leq 2$ for all integers $n \geq 1$.

Solution:
Define P(n) to be $1 \leq a_n \leq 2$. We prove P(n) holds for all integers $n \geq 1$ by strong induction.

Base Case P(1), P(2) Observe that $a_1 = a_2 = 1$, and $1 \leq 1 \leq 2$. So P(1) and P(2) hold.
Inductive Hypothesis: Suppose that P(j) is true for all $1 \leq j \leq k$ for some arbitrary integer $k \geq 2$.
Inductive Step:

$$\begin{aligned} a_{k+1} &= \frac{1}{2}(a_k + \frac{2}{a_{k-1}}) \\ &= \frac{a_k}{2} + \frac{1}{a_{k-1}}) \\ &\leq \frac{2}{2} + \frac{1}{a_{k-1}} && \text{By IH} \\ &\leq 1 + \frac{1}{1} && \text{By IH} \\ &= 2 \end{aligned}$$

$$\begin{aligned} a_{k+1} &= \frac{1}{2}(a_k + \frac{2}{a_{k-1}}) \\ &= \frac{a_k}{2} + \frac{1}{a_{k-1}}) \\ &\geq \frac{1}{2} + \frac{1}{a_{k-1}} && \text{By IH} \\ &\geq \frac{1}{2} + \frac{1}{2} && \text{By IH} \\ &= 1 \end{aligned}$$

So $1 \leq a_{k+1} \leq 2$.
Conclusion: Thus we have proven P(n) for all integers $n \geq 1$ by strong induction.

4. Modular Arithmetic [10 points]
(a) Prove or disprove: If $a \equiv b \pmod{10}$, then $a \equiv b \pmod{5}$. [5 points]

Solution:

True. Suppose that $a \equiv b \pmod{10}$. Then by definition of mod, $10 \mid (a - b)$. Then by definition of divides, there exists some integer k such that $a - b = 10k$ for some integer k. In particular, $a - b = 5(2k)$. Then $5 \mid (a - b)$. So $a \equiv b \pmod{5}$.

(b) Prove or disprove: If $a \equiv b \pmod{10}$, then $a \equiv b \pmod{20}$. [5 points]

Solution:

False. For example, for $a = 1$ and $b = 11$. Then $a \equiv b \pmod{10}$, but $a \not\equiv b \pmod{20}$.

5. That's Illegal [20 points]
Prove that the set of strings $\{0^n10^n : n \geq 0\}$ is not regular.

Solution:

$L = \{0^n10^n \; : \; n \geq 0\}$. Let D be an arbitrary DFA, and suppose for contradiction that D accepts L. Consider $S = \{0^n : n \geq 0\}$. Since S contains infinitely many strings and D has a finite number of states, two strings in S must end up in the same state. Say these strings are 0^i and 0^j for some integers $i, j \geq 0$ such that $i \neq j$. Append the string 10^i to both of these strings. The two resulting strings are:

$a = 0^i10^i$ Note that $a \in L$.

$b = 0^j10^i$ Note that $b \notin L$, since $i \neq j$.

Since a and b end up in the same state, but $a \in L$ and $b \notin L$, that state must be both an accept and reject state, which is a contradiction. Since D was arbitrary, there is no DFA that recognizes L, so L is not regular.