CSE 390Z: Mathematics for Computation Workshop

Practice 311 Final Solutions

Name: ________________________________

UW ID: ______________________________

Instructions:

- This is a simulated practice final. You will not be graded on your performance on this exam.
- This final was written to take 50 minutes. The real final will be 110 minutes.
- Nevertheless, please treat this as if it is a real exam. That means that you may not discuss with your neighbors, reference outside material, or use your devices during the next one hour period.
- If you get stuck on a problem, consider moving on and coming back later. In the actual exam, there will likely be opportunity for partial credit.
- There are 5 problems on this exam.
1. **Predicate Translation** [20 points]
Let the domain of discourse be people. Translate the following statements to predicate logic, using the following predicates:

- **City**($x$) := $x$ is a city
- **Suburb**($x$) := $x$ is a suburb
- **Town**($x$) := $x$ is a town
- **Person**($x$) := $x$ is a person
- **LivesIn**($x$, $y$) := $x$ lives in $y$
- **WorksIn**($x$, $y$) := $x$ works in $y$

You may also use $=$ and $\neq$ as predicates.

(a) (5 points) Alice lives in exactly one city.

**Solution:**
\[
\exists x \left( \text{City}(x) \land \text{LivesIn}(\text{Alice}, x) \land \forall y \left[ \left( \text{City}(y) \land \text{LivesIn}(\text{Alice}, y) \right) \rightarrow (y = x) \right] \right)
\]

OR
\[
\exists x \left( \text{City}(x) \land \text{LivesIn}(\text{Alice}, x) \land \forall y \left[ (y \neq x) \rightarrow \left( \neg \text{City}(y) \lor \neg \text{LivesIn}(\text{Alice}, y) \right) \right] \right)
\]

(b) (5 points) Everyone who lives in a suburb works in a city.

**Solution:**
\[
\forall x \left( \left( \exists y \left( \text{Suburb}(y) \land \text{LivesIn}(x, y) \right) \right) \rightarrow \exists z \left( \text{City}(z) \land \text{WorksIn}(x, z) \right) \right)
\]

(c) (5 points) There’s a town that no one lives in.

**Solution:**
\[
\exists x \left( \text{Town}(x) \land \neg \exists y \left( \text{Person}(y) \land \text{LivesIn}(y, x) \right) \right)
\]

OR
\[
\exists x \left( \text{Town}(x) \land \forall y \left( \text{Person}(y) \rightarrow \neg \text{LivesIn}(y, x) \right) \right)
\]

(d) (5 points) Every city is also a town, but not every town is a city.

**Solution:**
\[
\forall x \left( \text{City}(x) \rightarrow \text{Town}(x) \right) \land \exists x \left( \text{Town}(x) \land \neg \text{City}(x) \right)
\]

OR
\[
\forall x \left( \text{City}(x) \rightarrow \text{Town}(x) \right) \land \neg \forall x \left( \text{Town}(x) \rightarrow \text{City}(x) \right)
\]
2. **All the Machines!** [15 points]

Let the alphabet be $\Sigma = \{a, b\}$. Consider the language $L = \{w \in \Sigma^* : \text{every } a \text{ has a } b \text{ two characters later}\}$.

In other words, $L$ is the language of all strings in the alphabet $a, b$ where after any $a$, the character after the $a$ can be anything, but the character after that one must be a $b$.

Some strings in $L$ include $\varepsilon$, $abb$, $aabb$, $babb$. Some strings not in $L$ include $a$, $ab$, $aab$, $ababb$. Notice that the last two characters of the string cannot be an $a$.

(a) (5 points) Give a regular expression that represents $L$.

**Solution:**

$$(b \cup abb \cup aabb)^*$$

(b) (5 points) Give a CFG that represents $L$.

**Solution:**

$$S \rightarrow bS \mid aabbS \mid abbS \mid \varepsilon$$

(c) (5 points) Give a DFA that represents $L$.

**Solution:**

![DFA Diagram]
3. Induction [20 points]
Consider the following recursive definition of $a_n$:

$$
egin{align*}
  a_1 &= 1 \\
  a_2 &= 1 \\
  a_n &= \frac{1}{2} \left( a_{n-1} + \frac{2}{a_{n-2}} \right) & \text{for } n > 2
\end{align*}
$$

Prove that $1 \leq a_n \leq 2$ for all integers $n \geq 1$.

**Solution:**
Define $P(n)$ to be $1 \leq a_n \leq 2$. We prove $P(n)$ holds for all integers $n \geq 1$ by strong induction.

**Base Case** $P(1), P(2)$ Observe that $a_1 = a_2 = 1$, and $1 \leq 1 \leq 2$. So $P(1)$ and $P(2)$ hold.

**Inductive Hypothesis:** Suppose that $P(j)$ is true for all $1 \leq j \leq k$ for some arbitrary integer $k \geq 2$.

**Inductive Step:**

$$
\begin{align*}
  a_{k+1} &= \frac{1}{2} \left( a_k + \frac{2}{a_{k-1}} \right) \\
  &= \frac{a_k}{2} + \frac{1}{a_{k-1}} \\
  \leq & \frac{2}{2} + \frac{1}{a_{k-1}} & \text{By IH} \\
  \leq & 1 + \frac{1}{1} & \text{By IH} \\
  = & 2 \\
\end{align*}
$$

$$
\begin{align*}
  a_{k+1} &= \frac{1}{2} \left( a_k + \frac{2}{a_{k-1}} \right) \\
  &= \frac{a_k}{2} + \frac{1}{a_{k-1}} \\
  \geq & \frac{1}{2} + \frac{1}{a_{k-1}} & \text{By IH} \\
  \geq & \frac{1}{2} + \frac{1}{2} & \text{By IH} \\
  = & 1 \\
\end{align*}
$$

So $1 \leq a_{k+1} \leq 2$.

**Conclusion:** Thus we have proven $P(n)$ for all integers $n \geq 1$ by strong induction.
4. Modular Arithmetic [10 points]
(a) Prove or disprove: If $a \equiv b \pmod{10}$, then $a \equiv b \pmod{5}$. [5 points]

Solution:
True. Suppose that $a \equiv b \pmod{10}$. Then by definition of mod, $10 \mid (a - b)$. Then by definition of divides, there exists some integer $k$ such that $a - b = 10k$ for some integer $k$. In particular, $a - b = 5(2k)$. Then $5 \mid (a - b)$. So $a \equiv b \pmod{5}$.

(b) Prove or disprove: If $a \equiv b \pmod{10}$, then $a \equiv b \pmod{20}$. [5 points]

Solution:
False. For example, for $a = 1$ and $b = 11$. Then $a \equiv b \pmod{10}$, but $a \not\equiv b \pmod{20}$.
5. That's Illegal [20 points]
Prove that the set of strings \( \{0^n10^n : n \geq 0\} \) is not regular.

Solution:
\( L = \{0^n10^n : n \geq 0\} \). Let \( D \) be an arbitrary DFA, and suppose for contradiction that \( D \) accepts \( L \). Consider \( S = \{0^n : n \geq 0\} \). Since \( S \) contains infinitely many strings and \( D \) has a finite number of states, two strings in \( S \) must end up in the same state. Say these strings are \( 0^i \) and \( 0^j \) for some integers \( i, j \geq 0 \) such that \( i \neq j \). Append the string \( 10^i \) to both of these strings. The two resulting strings are:

\[
\begin{align*}
a &= 0^i10^i & \text{Note that } a \in L. \\
b &= 0^j10^i & \text{Note that } b \notin L, \text{ since } i \neq j.
\end{align*}
\]

Since \( a \) and \( b \) end up in the same state, but \( a \in L \) and \( b \notin L \), that state must be both an accept and reject state, which is a contradiction. Since \( D \) was arbitrary, there is no DFA that recognizes \( L \), so \( L \) is not regular.