CSE 390Z: Mathematics of Computing

Week 9 Workshop Solutions

Conceptual Review

Relations definitions: Let R be a relation on A. In other words, $R \subseteq A \times A$. Then:

- R is reflexive iff for all $a \in A$, $(a, a) \in R$.
- R is symmetric iff for all a, b, if $(a, b) \in R$, then $(b, a) \in R$.
- R is antisymmetric iff for all a, b, if $(a, b) \in R$ and $a \neq b$, then $(b, a) \notin R$.
- R is transitive iff for all a, b, if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

1. Context Free Grammars

Consider the following CFG which generates strings from the language V $:= \{0, 1, 2, 3, 4\}^*$

$$\mathbf{S} \to 0\mathbf{X}4$$
$$\mathbf{X} \to 1\mathbf{X}3 \mid 2$$

List 5 strings generated by the CFG and 5 strings from V not generated by the CFG. Then, summarize this CFG in your own words.

Solution:

Accepted:

- 024
- 01234
- 0112334
- 011123334
- 01111233334

Rejected:

- ϵ
- 2
- 0244
- 011234
- 10234

This CFG is all strings of the form $0\ 1^m\ 2\ 3^m\ 4$, where $m \geq 0$. That is, it's all strings made of one 0, followed by zero or more 1's, followed by a 2, followed by the same number of 3's as 1's, followed by one 4.

2. Constructing CFGs

For each of the following, construct a CFG for the specified language.

(a) Strings from the language $S := \{a\}^*$ with an even number of a's.

Solution:

$\mathbf{S} \to aa\mathbf{S} | \varepsilon$

(b) Strings from the language $S := \{a, b\}^*$ with odd length.

Solution:

$$\mathbf{S} \to \mathbf{CS}|a|b$$
$$\mathbf{C} \to aa\mathbf{C}|ab\mathbf{C}|ba\mathbf{C}|bb\mathbf{C}|\varepsilon$$

(c) Strings from the language $S := \{a, b\}^*$ with an even number of a's or an odd number of b's.

Solution:

$$\mathbf{S} \to \mathbf{E}|\mathbf{O}b\mathbf{O}$$
$$\mathbf{E} \to \mathbf{EE}|a\mathbf{E}a|b|\varepsilon$$
$$\mathbf{O} \to \mathbf{OO}|b\mathbf{O}b|a|\varepsilon$$

(d) Strings from the language $S := \{a, b\}^*$ with an equal number of a's and b's.

Solution:

$$\mathbf{S} \to a\mathbf{S}b\mathbf{S}|b\mathbf{S}a\mathbf{S}|\varepsilon$$

3. Relations Examples

(a) Consider the relation $R \subseteq \mathbb{Z} \times \mathbb{Z}$ defined by $(a, b) \in R$ iff $a \leq b + 1$. List 3 pairs of integers that are in R, and 3 pairs of integers that are not.

Solution:

In R: $(0, 0), (1, 0), (-1, 0)$
Not in R: $(2, 0), (3, 0), (17, 5)$

(b) Consider the relation $R \subseteq \mathbb{Z} \times \mathbb{Z}$ defined by $(a, b) \in R$ iff $a \leq b+1$. Determine if R is reflexive, symmetric, antisymmetric, and/or transitive. If a relation has a property, explain why. If not, state a counterexample.

Solution:

- Reflexive: Yes. For any integer a, it is true that $a \leq a + 1$. So $(a, a) \in R$.
- Symmetric: No. For example, $(0, 20) \in R$ but $(20, 0) \notin R$.
- Antisymmetric: No. For example $(0, 1) \in R$ and $(1, 0) \in R$.
- Transitive: No. For example $(2, 1) \in R$ and $(1, 0) \in R$, but $(2, 0) \notin R$.

4. Relations Proofs

Suppose that $R, S \subseteq \mathbb{Z} \times \mathbb{Z}$ are relations.

(a) Prove or disprove: If R and S are transitive, $R \cup S$ is transitive.

Solution:

False. Let $R = \{(1,2)\}$, $S = \{(2,1)\}$. By definition, R and S are transitive. By definition of union, $R \cup S = \{(1,2),(2,1)\}$. However, if $R \cup S$ was transitive, we would require $(1,1)$ to be in $R \cup S$, because (1,2) and (2,1) is in $R \cup S$. However, this is not the case. Therefore the claim is false.

(b) Prove or disprove: If R is symmetric, $\overline{R}$ (the complement of R) is symmetric.

Solution:

True. Since R is symmetric, we know the following.

$$\forall a \forall b\ [(a,b) \in R \to (b,a) \in R]$$

Taking the contrapositive, this is equivalent to:

$$\forall a \forall b\ [(b,a) \notin R \to (a,b) \notin R]$$

By the definition of complement, this is equivalent to:

$$\forall a \forall b\ [(b,a) \in \overline{R} \to (a,b) \in \overline{R}]$$

This is the definition of $\overline{R}$ being symmetric.