CSE 390Z: Mathematics of Computing

Week 7 Workshop Solutions

0. Strong Induction: Stamp Collection

A store sells 3 cent and 5 cent stamps. Use strong induction to prove that you can make exactly n cents worth of stamps for all $n \geq 10$.

Hint: you'll need multiple base cases for this - think about how many steps back you need to go for your inductive step.

Solution:

1 Let P(n) be defined as "You can buy exactly n cents of stamps". We will prove $P(n)$ is true for all integers $n \geq 10$ by strong induction.

2 **Base Cases** $(n = 10, 11, 12)$**:**

- $n = 10$: 10 cents of stamps can be made from two 5 cent stamps.
- $n = 11$: 11 cents of stamps can be made from one 5 cent and two 3 cent stamps.
- $n = 12$: 12 cents of stamps can be made from four 3 cent stamps.

3 **Inductive Hypothesis:** Suppose for some arbitrary integer $k \geq 12$, P(10) $\wedge$P(11) $\wedge$... $\wedge$P(k) holds.

4 **Inductive Step:**

$$\boxed{\textbf{Goal: } \text{Show } P(k+1), \text{ i.e. show that we can make } k+1 \text{ cents in stamps.}}$$

We want to buy $k + 1$ cents in stamps. By the I.H., we can buy exactly $(k + 1) - 3 = k - 2$ cents in stamps. Then, we can add another 3 cent stamp in order to buy $k+1$ packs of candy, so P$(k+1)$ is true.

Note: How did we decide how many base cases to have? Well, we wanted to be able to assume P$(k-2)$, and add 3 to achieve P$(k + 1)$. Therefore we needed to be able to assume that $k - 2 \geq 10$. Adding 2 to both sides, we needed to be able to assume that $k \geq 12$. So, we have to prove the base cases up to 12, that is: $10, 11, 12$.

Another way to think about this is that we had to use a fact from 3 steps back from $k + 1$ to $k - 2$ in the IS, so we needed 3 base cases.

5 So by strong induction, P(n) is true for all integers $n \geq 10$.

1. Strong Induction: Functions

Consider the function $f(n)$ defined for integers $n \geq 1$ as follows:
$f(1) = 1$ for $n = 1$
$f(2) = 4$ for $n = 2$
$f(3) = 9$ for $n = 3$
$f(n) = f(n-1) - f(n-2) + f(n-3) + 2(2n-3)$ for $n \geq 4$

Prove by strong induction that for all $n \geq 1$, $f(n) = n^2$.

Solution:

1 Let P(n) be defined as " $f(n) = n^2$". We will prove $P(n)$ is true for all integers $n \geq 1$ by strong induction.

2 **Base Cases** $(n = 1, 2, 3)$**:**

- $n = 1$: $f(1) = 1 = 1^2$.
- $n = 2$: $f(2) = 4 = 2^2$.
- $n = 3$: $f(3) = 9 = 3^2$

So the base cases hold.

3 **Inductive Hypothesis:** Suppose for some arbitrary integer $k \geq 3$, P(1) $\wedge$... $\wedge$P(k) hold.

4 **Inductive Step:**

Goal: Show $P(k+1)$, i.e. show that $f(k+1) = (k+1)^2$.

$$\begin{aligned}
f(k+1) &= f(k+1-1) - f(k+1-2) + f(k+1-3) + 2(2(k+1)-3) && \text{Definition of f} \\
&= f(k) - f(k-1) + f(k-2) + 2(2k-1) \\
&= k^2 - (k-1)^2 + (k-2)^2 + 2(2k-1) && \text{By IH} \\
&= k^2 - (k^2 - 2k + 1) + (k^2 - 4k + 4) + 4k - 2 \\
&= (k^2 - k^2 + k^2) + (2k - 4k + 4k) + (-1 + 4 - 2) \\
&= k^2 + 2k + 1 \\
&= (k+1)^2
\end{aligned}$$

So P$(k+1)$ holds.

5 **Conclusion:** So by strong induction, P(n) is true for all integers $n \geq 1$.

2. Strong Induction: Collecting Candy

A store sells candy in packs of 4 and packs of 7. Let P(n) be defined as "You are able to buy n packs of candy". For example, $P(3)$ is not true, because you cannot buy exactly 3 packs of candy from the store. However, it turns out that P(n) is true for any $n \geq 18$. Use strong induction on n to prove this.

Hint: you'll need multiple base cases for this - think about how many steps back you need to go for your inductive step.

Solution:

1 Let P(n) be defined as "You are able to buy n packs of candy". We will prove $P(n)$ is true for all integers $n \geq 18$ by strong induction.

2 **Base Cases** ($n = 18, 19, 20, 21$)**:**

- $n = 18$: 18 packs of candy can be made up of 2 packs of 7 and 1 pack of 4 ($18 = 2 * 7 + 1 * 4$).
- $n = 19$: 19 packs of candy can be made up of 1 pack of 7 and 3 packs of 4 ($19 = 1 * 7 + 3 * 4$).
- $n = 20$: 20 packs of candy can be made up of 5 packs of 4 ($20 = 5 * 4$).
- $n = 21$: 21 packs of candy can be made up of 3 packs of 7 ($21 = 3 * 7$).

3 **Inductive Hypothesis:** Suppose for some arbitrary integer $k \geq 21$, P(18) $\wedge$... $\wedge$P(k) hold.

4 **Inductive Step:**

Goal: Show $P(k+1)$, i.e. show that we can buy $k+1$ packs of candy.

We want to buy $k+1$ packs of candy. By the I.H., we can buy exactly $k-3$ packs, so we can add another pack of 4 packs in order to buy $k+1$ packs of candy, so P$(k+1)$ is true.

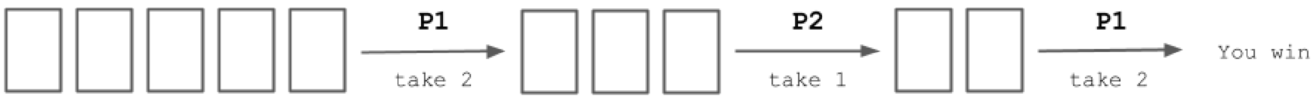
Note: How did we decide how many base cases to have? Well, we wanted to be able to assume P$(k-3)$, and add 4 to achieve P$(k+1)$. Therefore we needed to be able to assume that $k - 3 \geq 18$. Adding 3 to both sides, we needed to be able to assume that $k \geq 21$. So, we have to prove the base cases up to 21, that is: $18, 19, 20, 21$.

Another way to think about this is that we had to use a fact from 4 steps back from $k+1$ to $k-3$ in the IS, so we needed 4 base cases.

5 So by strong induction, P(n) is true for all integers $n \geq 18$.

3. Strong Induction: Cards on the Table

I've come up with a new card game that is played between 2 players as follows. We start with some integer $n \geq 1$ cards on the table. The two players then take turns removing cards from the table; in a single turn, a player can choose to remove either 1 or 2 cards from the table. A player wins by taking the last card. For example:



The person I've been playing with has been *very* careful about dealing the cards, and keeps winning; I think they know something I don't about this game. I want to use induction to prove that if $3|n$, the second player (P2) can guarantee a win, and if n is not divisible by 3, the first player (P1) can guarantee a win.

(a) How many base cases does this proof need? What should they be?

Solution:

3 base cases; 1, 2, and 3 cards.

(b) Use strong induction to prove that if $3|n$, P2 can guarantee a win, and if n is not divisible by 3, P1 can guarantee a win.

A note on the solution

There are two slightly different solutions below (Solution 1 and Solution 2). We made a video walk-through here of Solution 2.

Solution 1

Proof. Let Q(n) be defined as "if $3|n$, P2 can guarantee a win, and if n is not divisible by 3, P1 can guarantee a win". We will show Q(n) holds for all integers $n \geq 1$ by *strong* induction.

Base Cases:

- **n = 1:** P1 can take 1 card and win, and 1 is not divisible by 3, so Q(1) holds.
- **n = 2:** P1 can take 2 cards and win, and 2 is not divisible by 3, so Q(1) holds.
- **n = 3:** If P1 takes 1 card, P2 can take 2 cards and win. If P1 takes 2 cards, P2 can take 1 card and win. Since 3|3, Q(3) holds.

Inductive Hypothesis: Suppose for an arbitrary integer $k \geq 3$, Q$(1) \wedge \ldots \wedge$Q(k) hold.

Inductive Step:

- **case 1:** $3|k+1$
  By definition, $k+1 = 3x$ for some integer x. If P1 takes 1 card, P2 can take 2 cards, leaving $k-2 = 3(x-1)$ cards for the next round. If P1 takes 2 cards, P2 can take 1 card, also leaving $k-2 = 3(x-1)$ cards for the next round. Since $3|k-2$, by the Inductive Hypothesis P2 can win with $k-2$ cards, thus P2 can win with $k+1$ cards and Q$(k+1)$ holds.
- **case 2:** $k+1$ is not divisible by 3
  Either $k+1 \equiv_3 1$ or $k+1 \equiv_3 2$. If $k+1 \equiv_3 1$, then P1 can take 1 card, leaving k cards where $3|k$, thus $k = 3x$ for some integer x. If $k+1 \equiv_3 2$, then P1 can take 2 card, leaving k cards where $3|k$, thus $k = 3x$ for some integer x. P2 can then take 1 card or 2 cards, leaving $k-1 = 3x-1$ or $k-2 = 3x-2$ cards left, neither of which are divisible by 3. By the Inductive Hypothesis, P1 can always win with in either case, thus P1 can always win with $k+1$ cards and Q$(k+1)$ holds.

Thus, we have shown that Q(n) holds for all integers $n \geq 1$ by strong induction. □

Solution 2

Proof. Let Q(n) be defined as "if $3|n$, the next player can guarantee a win, and if n is not divisible by 3, the current player can guarantee a win". We will show Q(n) holds for all integers $n \geq 1$ by *strong* induction.

Base Cases:

- **n = 1:** The first (current) player can take 1 card and win, and 1 is not divisible by 3, so Q(1) holds.
- **n = 2:** The first (current) player can take 2 cards and win, and 2 is not divisible by 3, so Q(1) holds.
- **n = 3:** If the first (current) player takes 1 card, then the second (next) player can take 2 cards and win. If the first player takes 2 cards, the second player can take 1 card and win. Since 3|3, Q(3) holds.

Inductive Hypothesis: Suppose for an arbitrary integer $k \geq 3$, Q(1) $\wedge$... $\wedge$Q(k) hold.

Inductive Step:

- **case 1:** $3|k+1$
  By definition, $k+1 = 3x$ for some integer x. This is equivalent to $k-2 = 3x-3 = 3(x-1)$, so $3|k-2$. If the first player takes 1 card, then the next player will take 2 cards, leaving $k-2$ for the first player. If the first player were to instead take 2 cards, then the next player will take 1 card, leaving $k-2$ for the first player. Since $3|k-2$, by the Inductive Hypothesis, the second player will win with $k-2$ cards, thus they can win with $k+1$ cards and Q$(k+1)$ holds.
- **case 2:** $k+1$ is not divisible by 3
  Either $k+1 \equiv_3 1$ or $k+1 \equiv_3 2$. If $k+1 \equiv_3 1$, then the first player can take 1 card, leaving k cards where $3|k$. If $k+1 \equiv_3 2$, then the first player can take 2 cards, leaving $k-1$ cards where $3|(k-1)$. Either way, the second player will have to play when the number of cards left on the table is divisible by 3. By the Inductive Hypothesis, the player after the second player (the first player in this case) can always win in either case, so the first player can always win with $k+1$ cards and Q$(k+1)$ holds.

Thus, we have shown that Q(n) holds for all integers $n \geq 1$ by strong induction.

Notice how in the second proof, we defined Q(n) slightly differently. Instead of explicitly stating P1 and P2 in our definition, we used "the current player" and "the next player". How does this make our proof different (and arguably simpler)? □

4. How does mod work again?

In 311, you learned the mathematical definition of mod. Here is one recursive implementation of mod:

```
// returns the result of a mod m
public static int modr(int a, int m) {
    if (a < m) {
        return a;
    } else {
        return modr(a - m, m);
    }
}
```

Use strong induction to prove that given an arbitrary positive integer m, my `modr` method correctly returns the result of a mod m for any non-negative integer a (i.e. $a \in \mathbb{Z}$, $a \geq 0$).

Solution:

1 Let P(n) be defined as `modr(n, m)` returns n mod m for an arbitrary positive integer m. We will prove P(n) for all non-negative integers n using strong induction.

2 **Base Case:** $0 \leq n < m$
If $0 \leq n < m$, then $n = m \cdot 0 + n$, and thus n mod $m = n$, which is what `modr(n,m)` will return, thus P(n) holds.

3 **Inductive Hypothesis:** Suppose P(1) $\wedge$P(2) $\wedge$... $\wedge$P(k) holds for an arbitrary integer $k \geq m - 1$.

4 **Inductive Step:**

> **Goal:** Show $P(k+1)$, i.e. show that `modr(k+1,m)` returns $(k+1)$ mod m.

By the division theorem, we know that $k+1 = q \cdot m + r$, where $r = (k+1)$ mod m. We can subtract m from both sides of this equation to get $k+1-m = (q-1) \cdot m + r$, which means $r = (k+1)$ mod $m =$ $(k+1-m)$ mod m (note that the mod method makes a recursive call for mod$(k+1-m, m)$ in this case). m is defined to be an arbitrary positive integer, i.e. $m \geq 1$, so then $k+1-m \leq k$. By the inductive hypothesis, we know that `modr(k+1−m, m)` returns $(k+1-m)$ mod $m = (k+1)$ mod m. Since `modr(k+1, m)` returns the value returned by `modr(k+1−m, m)` and `modr(k+1−m, m)` returns $(k+1)$ mod m, `modr(k+1, m)` returns $(k+1)$ mod m and thus P$(k=1)$ holds.

5 Thus by strong induction, P(n) holds for all non-negative integers n.

5. Recursively Defined Sets

Write a recursive definition of a set of integers satisfying the given properties.

(a) All integers x satisfying $x \equiv 2 \pmod 5$.

Solution:

Basis Case: $2 \in S$
Recursive Case: If $x \in S$, then $x + 5 \in S$ and $x - 5 \in S$

(b) The set of coordinates where the x-value is an index, and the y-value is the Fibonacci number at that index. I.e. the coordinates $(0, 0) \in S$, $(1, 1) \in S$, $(2, 1) \in S$, $(3, 2) \in S$, etc.

Solution:

Basis Case: $(0, 0) \in S$ and $(1, 1) \in S$
Recursive Case: if $(n - 1, x) \in S$ and $(n, y) \in S$ then $(n + 1, x + y) \in S$

6. Recursively Defined Sets Counterexamples

Find a counterexample for each of the following claims.

(a) Every recursively defined set of integers has infinitely many elements.

Solution:

- Universe: Integers
- Base case: $0, 1 \in R$
- Recursive step: If $x, y \in R$, then $|x - y| \in R$.

(b) If R is a recursively defined set of integers with infinitely many elements, then there is some integer m such that every integer greater than m is in R.

Solution:

- Universe: Integers
- Base case: $0, 2 \in R$
- Recursive step: If $x, y \in R$, then $x + y \in R$.