CSE 390Z: Mathematics for Computation Workshop

Week 4 Workshop Solutions

Conceptual Review

(a) Translate "all cats are friends with a dog" to predicate logic. Domain of dicourse: mammals.

Solution:

$\forall x(\text{Cat}(x) \rightarrow \exists y(\text{Dog}(y) \wedge \text{Friends}(x, y)))$

(b) Inference Rules:

Introduce $\vee$:	$\frac{A}{\therefore\ A \vee B,\ B \vee A}$	Eliminate $\vee$:	$\frac{A \vee B\ ;\ \neg A}{\therefore\ B}$
Introduce $\wedge$:	$\frac{A\ ;\ B}{\therefore\ A \wedge B}$	Eliminate $\wedge$:	$\frac{A \wedge B}{\therefore\ A\ ,\ B}$
Direct Proof:	$\frac{A \Rightarrow B}{\therefore\ A \rightarrow B}$	Modus Ponens:	$\frac{A\ ;\ A \rightarrow B}{\therefore\ B}$
Intro $\exists$:	$\frac{P(c) \text{ for some } c}{\therefore\ \exists x P(x)}$	Eliminate $\exists$:	$\frac{\exists x P(x)}{\therefore\ P(c) \text{ for a fresh } c}$
Intro $\forall$:	$\frac{P(a);\ a \text{ is arbitrary}}{\therefore\ \forall x P(x)}$	Eliminate $\forall$:	$\frac{\forall x P(x)}{\therefore\ P(a);\ a \text{ is arbitrary}}$

(c) What are DeMorgan's Laws for Quantifiers?

Solution:

¬∀x P(x) ≡∃x ¬P(x)
¬∃x P(x) ≡∀x ¬P(x)

(d) Given $A \wedge B$, prove $A \vee B$

Solution:

1. $A \wedge B$ (Given)
2. A (Elim $\wedge$: 1.)
3. $A \vee B$ (Intro $\vee$: 2.)

Given $P \rightarrow R$, $R \rightarrow S$, prove $P \rightarrow S$.

1. $P \rightarrow R$ (Given)
2. $R \rightarrow S$ (Given)
   3.1 P (Assumption)
   3.2 R (Modus Ponens: 3.1, 1)
   3.3 S (Modus Ponens: 3.2, 2)
3. $P \rightarrow S$ (Direct Proof Rule; 3.1-3.3)

(e) How do you prove a "for all" statement? E.g. prove $\forall x P(x)$. How do you prove a "there exists" statement? E.g. prove $\exists x P(x)$.

Solution:

To prove "for all", we show that for any **arbitrary** a in the domain, $P(a)$ holds. To prove "there exists", we show that for some specific a in the domain, $P(a)$ holds.

1. Tricky Translations

Translate the following English sentences to predicate logic. The domain is integers, and you may use $=$, $\neq$, and $>$ as predicates. Assume the predicates Prime, Composite, and Even have been defined appropriately.

(a) 2 is prime.

Solution:

Prime(2)

(b) Every **positive** integer is prime or composite, but not both.

Solution:

$\forall x\ ((x > 0) \rightarrow (\text{Prime}(x) \oplus \text{Composite}(x)))$

OR

$\forall x\ ((x > 0) \rightarrow [(\text{Prime}(x) \land \neg\text{Composite}(x)) \lor (\neg\text{Prime}(x) \land \text{Composite}(x))])$

(c) There is **exactly one** even prime.

Solution:

$\exists x((\text{Even}(x) \land \text{Prime}(x) \land \forall y[(\text{Even}(y) \land \text{Prime}(y)) \rightarrow (y = x)])$

OR

$\exists x((\text{Even}(x) \land \text{Prime}(x) \land \forall y[(y \neq x) \rightarrow \neg(\text{Even}(y) \land \text{Prime}(y))])$

(d) 2 is the only even prime.

Solution:

$\forall x\ ((x = 2) \leftrightarrow \text{Prime}(x) \land \text{Even}(x))$

(e) Some, but not all, composite integers are even.

Solution:

$\exists x(\text{Composite}(x) \land \text{Even}(x)) \land \neg\forall x(\text{Composite}(x) \rightarrow \text{Even}(x))$

OR

$\exists x(\text{Composite}(x) \land \text{Even}(x)) \land \exists x(\text{Composite}(x) \land \neg\text{Even}(x))$

2. Propositional Proof 1

(a) Prove that given $p \rightarrow q$, $\neg s \rightarrow \neg q$, and p, we can conclude s.

Solution:

1. $p \rightarrow q$ (Given)
2. $\neg s \rightarrow \neg q$ (Given)
3. p (Given)

4. q (Modus Ponens; 3,1)
5. $q \to s$ (Contrapositive; 2)
6. s (Modus Ponens; 4,5)

(b) Prove that given $\neg(p \lor q) \to s$, $\neg p$, and $\neg s$, we can conclude q.

Solution:

1. $\neg(p \lor q) \to s$ (Given)
2. $\neg p$ (Given)
3. $\neg s$ (Given)
4. $\neg s \to \neg\neg(p \lor q)$ (Contrapositive; 1)
5. $\neg s \to (p \lor q)$ (Double Negation; 4)
6. $p \lor q$ (Modus Ponens; 3,5)
7. q (Elim $\lor$; 6,2)

3. Propositional Proofs 2

(a) Prove that given $p \to q$, we can conclude $(p \land r) \to q$

Solution:

1. $p \to q$ (Given)
   2.1 $p \land r$ (Assumption)
   2.2 p (Elim $\land$; 2.1)
   2.3 q (Modus Ponens; 2.2, 1.)
2. $(p \land r) \to q$ (Direct proof rule; 2.1-2.3)

(b) Prove that given $p \lor q$, $q \to r$, and $r \to s$, we can conclude $\neg p \to s$.

Solution:

1. $p \lor q$ (Given)
2. $q \to r$ (Given)
3. $r \to s$ (Given)
   4.1 $\neg p$ (Assumption)
   4.2 q (Elim $\lor$; 1, 4.1)
   4.3 r (Modus Ponens; 4.2, 2)
   4.4 s (Modus Ponens; 4.3, 3)
4. $\neg p \to s$ (Direct proof rule; 4.1-4.4)

4. Predicate Proofs 1

(a) Prove that $\forall x P(x) \to \exists x P(x)$. You may assume that the domain is nonempty.

Solution:

1.1. $\forall x P(x)$ (Assumption)

1.2. $P(a)$ (Elim $\forall$: 1.1)

1.3. $\exists x P(x)$ (Intro $\exists$: 1.2)

1. $\forall x P(x) \rightarrow \exists x P(x)$ (Direct Proof Rule, from 1.1-1.3)

(b) Given $\forall x(T(x) \rightarrow M(x))$ and $\exists x(T(x))$, prove that $\exists x(M(x))$.

Solution:

1. $\forall x(T(x) \rightarrow M(x))$ (Given)
2. $\exists x(T(x))$ (Given)
   Let r be the object that satisfies $T(r)$
3. $T(r)$ ($\exists$ elimination, from 2)
4. $T(r) \rightarrow M(r)$ ($\forall$ elimination, from 1)
5. $M(r)$ (Modus ponens, from 3 and 4)
6. $\exists x(M(x))$ ($\exists$ introduction, from 5)

(c) Given $\forall x(P(x) \rightarrow Q(x))$, prove that $\exists x P(x) \rightarrow \exists y Q(y)$. You may assume that the domain is non-empty.

Solution:

1. $\forall x(P(x) \rightarrow Q(x))$ (Given)

2.1. $\exists x(P(x))$ (Assumption)

Let r be the object that satisfies $P(r)$

2.2. $P(r)$ ($\exists$ elimination, from 2.1)

2.3. $P(r) \rightarrow Q(r)$ ($\forall$ elimination, from 1)

2.4. $Q(r)$ (Modus Ponens, from 2.2 and 2.3)

2.5. $\exists y(Q(y))$ ($\exists$ introduction, from 2.4)

2. $\exists x P(x) \rightarrow \exists y Q(y)$ (Direct Proof Rule, from 2.1-2.5)

5. Negating Predicates

Last week, we translated the sentence "Not every actor has been featured in a movie" to predicate logic. The domain of discourse was movies and actors, and we had the following predicates: Movie(x) ::= x is a movie, Actor(x) ::= x is an actor, Features(x, y) ::= x features y.

This was Timothy's translation: $\neg\forall x(\text{Actor}(x) \rightarrow \exists y(\text{Movie}(y) \wedge \text{Features}(y, x)))$

This was Cade's translation: $\exists x(\text{Actor}(x) \wedge \forall y(\text{Movie}(y) \rightarrow \neg\text{Features}(y, x)))$

(a) Adam claims that Timothy and Cade are both correct. Do you agree with Adam?

Solution:

Yes, both translations are correct.

(b) Use a chain of predicate logic equivalences to prove that Timothy and Cade's translations are equivalent. **Hint:** You may wish to use De Morgan's Law for Predicates and the Law of Implications.

Solution:

$\neg\forall x(\text{Actor}(x) \to \exists y(\text{Movie}(y) \land \text{Features}(y,x)))$	
$\equiv \exists x(\neg(\text{Actor}(x) \to \exists y(\text{Movie}(y) \land \text{Features}(y,x))))$	DeMorgan's Law for Predicates
$\equiv \exists x(\neg(\neg\text{Actor}(x) \lor \exists y(\text{Movie}(y) \land \text{Features}(y,x))))$	Law of Implications
$\equiv \exists x(\neg\neg\text{Actor}(x) \land \neg\exists y(\text{Movie}(y) \land \text{Features}(y,x))))$	DeMorgan's Law
$\equiv \exists x(\text{Actor}(x) \land \neg\exists y(\text{Movie}(y) \land \text{Features}(y,x))))$	Double Negation
$\equiv \exists x(\text{Actor}(x) \land \forall y(\neg(\text{Movie}(y) \land \text{Features}(y,x)))))$	DeMorgan's Law for Predicates
$\equiv \exists x(\text{Actor}(x) \land \forall y(\neg\text{Movie}(y) \lor \neg\text{Features}(y,x)))))$	DeMorgan's Law
$\equiv \exists x(\text{Actor}(x) \land \forall y(\text{Movie}(y) \to \neg\text{Features}(y,x)))))$	Law of Implications

6. Predicate Proofs 2

Given $\forall x\ (P(x) \lor Q(x))$ and $\forall y\ (\neg Q(y) \lor R(y))$, prove $\exists x\ (P(x) \lor R(x))$. You may assume that the domain is not empty.

Solution:

1.	$\forall x\ (P(x) \lor Q(x))$	[Given]
2.	$\forall y\ (\neg Q(y) \lor R(y))$	[Given]
3.	$P(a) \lor Q(a)$	[Elim $\forall$: 1]
4.	$\neg Q(a) \lor R(a)$	[Elim $\forall$: 2]
5.	$Q(a) \to R(a)$	[Law of Implication: 4]
6.	$\neg\neg P(a) \lor Q(a)$	[Double Negation: 3]
7.	$\neg P(a) \to Q(a)$	[Law of Implication: 5]
	8.1. $\neg P(a)$ [Assumption]	
	8.2. $Q(a)$ [Modus Ponens: 8.1, 7]	
	8.3. $R(a)$ [Modus Ponens: 8.2, 5]	
8.	$\neg P(a) \to R(a)$	[Direct Proof]
9.	$\neg\neg P(a) \lor R(a)$	[Law of Implication: 8]
10.	$P(a) \lor R(a)$	[Double Negation: 9]
11.	$\exists x\ (P(x) \lor R(x))$	[Intro $\exists$: 10]

7. Predicate Proofs 3

Write a formal proof to show: If n, m are odd, then $n + m$ is even.
Let the predicates Odd(x) and Even(x) be defined as follows where the domain of discourse is integers:

$$\text{Odd}(x) := \exists y\ (x = 2y + 1)$$
$$\text{Even}(x) := \exists y\ (x = 2y)$$

Solution:

1.	Let x be an arbitrary integer.		
2.	Let y be an arbitrary integer.		
	3.1.	Odd$(x) \wedge$ Odd(y)	[Assumption]
	3.2.	Odd(x)	[Elim $\wedge$: 3.1]
	3.3.	$\exists k\ (x = 2k + 1)$	[Definition of Odd, 3.2]
	3.4.	$x = 2k + 1$	[Elim $\exists$: 3.3]
	3.5.	Odd(y)	[Elim $\wedge$: 3.1]
	3.6.	$\exists k\ (y = 2k + 1)$	[Definition of Odd, 3.5]
	3.7.	$y = 2j + 1$	[Elim $\exists$: 3.7]
	3.8.	$x + y = 2k + 1 + 2j + 1$	[Algebra: 3.4, 3.7]
	3.9.	$x + y = 2(k + j + 1)$	[Algebra: 3.8]
	3.10.	$\exists r\ (x + y = 2r)$	[Intro $\exists$: 3.9]
	3.11.	Even$(x + y)$	[Definition of Even, 3.10]
3.	Odd$(x) \wedge$ Odd$(y) \rightarrow$ Even$(x + y)$		[Direct Proof Rule]
4.	$\forall m($Odd$(x) \wedge$ Odd$(m) \rightarrow$ Even$(x + m))$		[Intro $\forall$: 2,3]
5.	$\forall n \forall m($Odd$(n) \wedge$ Odd$(m) \rightarrow$ Even$(n + m))$		[Intro $\forall$: 1,4]

8. Challenge: Predicate Negation

Translate "You can fool all of the people some of the time, and you can fool some of the people all of the time, but you can't fool all of the people all of the time" into predicate logic. Then, negate your translation. Then, translate the negation back into English.

Hint: Let the domain of discourse be all people and all times, and let $P(x, y)$ be the statement "You can fool person x at time y". You can get away with not defining any other predicates if you use P.

Solution:

The original statement can thus be translated as

$$(\forall x \exists y P(x, y)) \wedge (\exists z \forall a P(z, a)) \wedge (\neg \forall b \forall c P(b, c))$$

The negation of this statement, in predicate logic, is

$$(\exists x \forall y \neg P(x, y)) \vee (\forall z \exists a \neg P(z, a)) \vee (\forall b \forall c P(b, c))$$

which in English translates to

"There are some people you can't ever fool, or all people have some time at which you can't fool them, or you can fool everyone at all times"

9. Challenge: Formal Proof

Use a formal proof to show that for any propositions a, b, c, the following holds.

$$a \to (b \to (c \to ((a \wedge b) \wedge c)))$$

Solution:

Left as an exercise. General idea is to introduce several assumptions, followed by several intro $\wedge$ steps, followed by several applications of the Direct Proof Rule.