CSE 390Z: Mathematics for Computation Workshop

Practice 311 Midterm 2 Solutions

Name: ______________________________

UW ID: ______________________________

Instructions:

- This is a **simulated practice midterm**. You will **not** be graded on your performance on this exam.
- Nevertheless, please treat this as if it is a real exam. That means that you may not discuss with your neighbors, reference outside material, or use your devices during the next 50 minute period.
- If you get stuck on a problem, consider moving on and coming back later. In the actual exam, there will likely be opportunity for partial credit.
- There are 3 problems on this exam, totaling 60 points.

1. Predicate Translation [20 points]
Let the domain of discourse be people. Translate the following statements to predicate logic, using the following predicates:

Student(x) := x is a student
Professor(x) := x is a professor
EnrolledInClass(x, y) := x is enrolled a class taught by y

You may also use $=$ and $\neq$ as predicates.

(a) (5 points) Every professor has at least one student enrolled in their class.

Solution:

$\forall x(\text{Professor}(x) \rightarrow \exists y(\text{Student}(y) \wedge \text{EnrolledInClass}(y, x)))$

(b) (5 points) There's a student who is enrolled in two different professors' classes.

Solution:

$\exists x(\text{Student}(x) \wedge \exists y(\text{Professor}(y) \wedge \text{EnrolledInClass}(x, y)) \wedge \exists z(\text{Professor}(z) \wedge (z \neq y) \wedge \text{EnrolledInClass}(x, z)))$

(c) (5 points) There's a professor who is enrolled in another professor's class.

Solution:

$\exists x \exists y(\text{Professor}(x) \wedge \text{Professor}(y) \wedge \text{EnrolledInClass}(x, y))$

(d) (5 points) All students are enrolled in some class, taught by some professor. (May not be the same professor for everyone).

Solution:

$\forall x(\text{Student}(x) \rightarrow \exists y(\text{Professor}(y) \wedge \text{EnrolledInClass}(x, y)))$

2. Set Proof [20 points]
Suppose that for sets A, B, C, the facts $A \subseteq B$ and $B \subseteq C$ are given. Write an English proof to show that $B \times A \subseteq C \times C$.

Solution:
Suppose that for sets A, B, C, we have $A \subseteq B$ and $B \subseteq C$ (these are our givens). Let $x \in B \times A$ be arbitrary. Then by definition of Cartesian Product, $x = (y, z)$ for $y \in B$ and $z \in A$. Then since $y \in B$ and $B \subseteq C$, $y \in C$. Similarly since $z \in A$ and $A \subseteq B$, $z \in B$. Then since $z \in B$ and $B \subseteq C$, we have $z \in C$. Therefore we have shown that $y \in C$ and $z \in C$. Then by definition of Cartesian Product, $x \in C \times C$. Since x was arbitrary, we have shown $B \times A \subseteq C \times C$.

3. Induction [20 points]
Prove by induction on n that for all integers $n \geq 0$ the inequality $(3+\pi)^n \geq 3^n + n\pi 3^{n-1}$ is true.

Solution:

1. Let $P(n)$ be "$(3+\pi)^n \geq 3^n + n\pi 3^{n-1}$". We will prove $P(n)$ is true for all $n \in \mathbb{N}$, by induction.
2. **Base case** (n = 0)**:** $(3+\pi)^0 = 1$ and $3^0 + 0 \cdot \pi \cdot 3^{-1} = 1$, since $1 \geq 1$, $P(0)$ is true.
3. **Inductive Hypothesis:** Suppose that $P(k)$ is true for some arbitrary integer $k \in \mathbb{N}$.
4. **Inductive Step:**

Goal: Show $P(k+1)$, i.e. show $(3+\pi)^{k+1} \geq 3^{k+1} + (k+1)\pi 3^{(k+1)-1} = 3^{k+1} + (k+1)\pi 3^k$

$$\begin{aligned}
(3+\pi)^{k+1} &= (3+\pi)^k \cdot (3+\pi) && \text{(Factor out } (3+\pi)\text{)} \\
&\geq (3^k + k3^{k-1}\pi) \cdot (3+\pi) && \text{(By I.H., } (3+\pi) \geq 0\text{)} \\
&= 3 \cdot 3^k + 3^k\pi + 3k3^{k-1}\pi + k3^{k-1}\pi^2 && \text{(Distributive property)} \\
&= 3^{k+1} + 3^k\pi + k3^k\pi + k3^{k-1}\pi^2 && \text{(Simplify)} \\
&= 3^{k+1} + (k+1)3^k\pi + k3^{k-1}\pi^2 && \text{(Factor out } (k+1)\text{)} \\
&\geq 3^{k+1} + (k+1)\pi 3^k && (k3^{k-1}\pi^2 \geq 0)
\end{aligned}$$

5. So by induction, $P(n)$ is true for all $n \in \mathbb{N}$.