

CSE 390Z: Mathematics for Computation Workshop

Practice 311 Midterm 1 Solutions

Name: _____

UW ID: _____

Instructions:

- This is a **simulated practice midterm**. You will **not** be graded on your performance on this exam.
- Nevertheless, please treat this as if it is a real exam. That means that you may not discuss with your neighbors, reference outside material, or use your devices during the next 50 minute period.
- If you get stuck on a problem, consider moving on and coming back later. In the actual exam, there will likely be opportunity for partial credit.
- There are 3 problems on this exam, totaling 55 points.

1. Predicate Translation [15 points]

Let the domain of discourse be novels, comic books, movies, and TV shows. Translate the following statements to predicate logic, using the following predicates:

$\text{Novel}(x) := x$ is a novel

$\text{Comic}(x) := x$ is a comic book

$\text{Movie}(x) := x$ is a movie

$\text{Show}(x) := x$ is a TV show

$\text{Adaptation}(x, y) := x$ is an adaptation of y

(a) (5 points) A novel cannot be adapted into both a movie and a TV show.

Solution:

$$\forall x(\text{Novel}(x) \rightarrow \forall m \forall s((\text{Movie}(m) \wedge \text{Show}(s)) \rightarrow \neg(\text{Adaptation}(m, x) \wedge \text{Adaptation}(s, x))))$$

(b) (5 points) Every movie is an adaptation of a novel or a comic book.

Solution:

$$\forall m(\text{Movie}(m) \rightarrow \exists x(\text{Adaptation}(m, x) \wedge (\text{Novel}(x) \vee \text{Comic}(x))))$$

(c) (5 points) Every novel has been adapted into exactly one movie.

Solution:

$$\forall x(\text{Novel}(x) \rightarrow \exists m(\text{Movie}(m) \wedge \text{Adaptation}(m, x) \wedge \forall n((\text{Movie}(n) \wedge (n \neq m)) \rightarrow \neg \text{Adaptation}(n, x))))$$

OR

$$\forall x(\text{Novel}(x) \rightarrow \exists m(\text{Movie}(m) \wedge \text{Adaptation}(m, x) \wedge \forall n(\text{Adaptation}(n, x) \rightarrow (\neg \text{Movie}(n) \vee n = m))))$$

OR

$$\forall x(\text{Novel}(x) \rightarrow \exists m(\text{Movie}(m) \wedge \text{Adaptation}(m, x) \wedge \forall n((\text{Adaptation}(n, x) \wedge \text{Movie}(n)) \rightarrow (n = m))))$$

*Note that a great exercise is to show that the above 3 solutions are all logically equivalent :)

2. Number Theory Proof [20 points]

Recall this definition of odd: $\text{Odd}(x) := \exists y(x = 2y + 1)$. Write an English proof to show that for all odd integers k , the statement $8 \mid k^2 - 1$ holds.

Hint: At some point in your proof, you'll need to show that for any integer a , $a(a + 1)$ is even. When you reach this point, feel free to break your proof up into the case where a is even, and the case where a is odd.

Solution:

Let k be an arbitrary odd integer. Then $k = 2a + 1$ for some integer a . Then $k^2 - 1 = (2a + 1)^2 - 1 = 4a^2 + 4a + 1 - 1 = 4a^2 + 4a = 4a(a + 1)$.

Consider the case where a is odd. Then $a = 2b + 1$ for some integer b . Then $k^2 - 1 = 4a(a + 1) = 4(2b + 1)(2b + 2) = 8(2b + 1)(b + 1)$. By closure of integers under multiplication and addition, $k^2 - 1 = 8c$ for an integer c . Thus in this case, $8 \mid k^2 - 1$.

Consider the case where a is even. Then $a = 2b$ for some integer b . Then $k^2 - 1 = 4a(a + 1) = 4(2b)(2b + 1) = 8b(2b + 1)$. By closure of integers under multiplication and addition, $k^2 - 1 = 8c$ for an integer c . Thus in this case, $8 \mid k^2 - 1$.

So in all cases, $8 \mid k^2 - 1$. Since k was an arbitrary odd integer, we have proved the claim.

3. Induction [20 points]

Prove by induction that $3^n - 1$ is divisible by 2 for any integer $n \geq 1$.

Solution:

1. Let $P(n)$ be the statement " $3^n - 1$ is divisible by 2". We prove $P(n)$ for all integers $n \geq 1$ by induction.
2. Base Case: When $n = 1$, $3^n - 1 = 3^1 - 1 = 3 - 1 = 2$. Since $2 \mid 2$, the base case holds.
3. Inductive Hypothesis: Suppose that $P(k)$ holds for some arbitrary integer $k \geq 1$. Then $2 \mid 3^k - 1$. Then by definition of divides, there exists some integer a such that $3^k - 1 = 2a$.
4. Inductive Step: Observe that...

$3^{k+1} - 1 = 3(3^k) - 1$	Definition of Exponent
$= 3(3^k - 1 + 1) - 1$	Subtract and Add by 1
$= 3(2a + 1) - 1$	By IH
$= 6a + 3 - 1$	Algebra
$= 6a + 2$	Algebra
$= 2(3a + 1)$	Algebra

Thus by definition of divides, $2 \mid 3^{k+1} - 1$. So $P(k + 1)$ holds.

5. Thus we have proven $P(n)$ for all integers $n \geq 1$ by induction.