CSE 390Z: Mathematics for Computation Workshop

Practice 311 Final Solutions

Name: ______________________________

UW ID: ______________________________

Instructions:

- This is a **simulated practice final**. You will **not** be graded on your performance on this exam.
- This final was written to take 50 minutes. The real final will be an hour and 50 minutes.
- Nevertheless, please treat this as if it is a real exam. That means that you may not discuss with your neighbors, reference outside material, or use your devices during the next 50 minute period.
- If you get stuck on a problem, consider moving on and coming back later. In the actual exam, there will likely be opportunity for partial credit.
- There are 5 problems on this exam.

1. All the Machines! [15 points]
Let the alphabet be $\Sigma = \{a, b\}$. Consider the language $L = \{w \in \Sigma^* : \text{every } a \text{ has a } b \text{ two characters later}\}$. In other words, L is the language of all strings in the alphabet a, b where after any a, the character after the a can be anything, but the character after that one must be a b.

Some strings in L include ε, abb, $aabb$, $bbbbabb$. Some strings not in L include a, ab, aab, $ababb$. Notice that the last two characters of the string cannot be an a.

(a) (5 points) Give a regular expression that represents L.

Solution:

$(b \cup abb \cup aabb)^*$

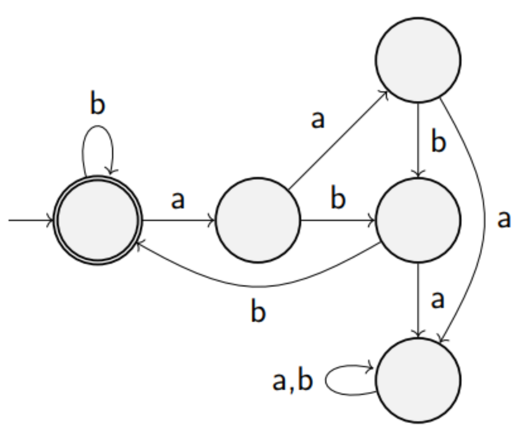
(b) (5 points) Give a CFG that represents L.

Solution:

S$\rightarrow$ b**S** | $aabb$**S** | abb**S** | ε

(c) (5 points) Give a DFA that represents L.

Solution:



2. Induction 1 [20 points]
Recall the recursive definition of a list of integers:

- [ ] is the empty list
- If L is a list and a is an integer, then $a :: L$ is a list whose first element is a, followed by the elements of L.

Consider the following functions defined on lists:
len([ ]) $= 0$
len($x :: L$) $= 1+$ len(L)

inc([ ]) $=$ [ ]
inc($x :: L$) $= (x+1) ::$ inc(L)

sum([ ]) $= 0$
sum($x :: L$) $= x +$ sum(L)

Prove that for all lists L, sum(inc(L)) $=$ sum(L) $+$ len(L).

Solution:
Let P(L) be "sum(inc(L)) $=$ sum(L) $+$ len(L)". We prove that P(L) is true for all lists L by structural induction.
Base Case: $L =$ [ ]. Then:

$\text{sum}(\text{inc}([\,]))$	$= \text{sum}([\,])$	Definition of inc
	$= 0$	Definition of sum
	$= 0 + 0$	Algebra
	$= \text{sum}([\,]) + \text{len}([\,])$	Definition of sum, len

Inductive Hypothesis: Suppose that P(L) is true for an arbitrary list L.
Inductive Step: We aim to show that P($x :: L$) holds.

$\text{sum}(\text{inc}(x :: L))$	$= \text{sum}((x+1) :: \text{inc}(L))$	Definition of inc
	$= (x+1) + \text{sum}(\text{inc}(L))$	Definition of sum
	$= (x+1) + \text{sum}(L) + \text{len}(L)$	Inductive Hypothesis
	$= x + \text{sum}(L) + 1 + \text{len}(L)$	Algebra
	$= \text{sum}(x :: L) + \text{len}(x :: L)$	Definition of sum, len

So P($x :: L$) holds.

Conclusion: Thus P(L) holds for all lists L by structural induction.

3. Induction 2 [20 points]
Consider the following recursive definition of a_n:

$$a_1 = 1$$
$$a_2 = 1$$
$$a_n = \frac{1}{2}(a_{n-1} + \frac{2}{a_{n-2}}) \qquad \text{for } n > 2$$

Prove that $1 \leq a_n \leq 2$ for all integers $n \geq 1$.

Solution:
Define P(n) to be $1 \leq a_n \leq 2$. We prove P(n) holds for all integers $n \geq 1$ by strong induction.

Base Case P(1), P(2) Observe that $a_1 = a_2 = 1$, and $1 \leq 1 \leq 2$. So P(1) and P(2) hold.
Inductive Hypothesis: Suppose that P(j) is true for all $1 \leq j \leq k$ for some arbitrary integer $k \geq 2$.
Inductive Step:

$$\begin{aligned} a_{k+1} &= \frac{1}{2}(a_k + \frac{2}{a_{k-1}}) \\ &= \frac{a_k}{2} + \frac{1}{a_{k-1}} \\ &\leq \frac{2}{2} + \frac{1}{a_{k-1}} && \text{By IH, since } a_k \leq 2 \\ &\leq 1 + \frac{1}{1} && \text{By IH, since } a_{k-1} \geq 1 \text{, so } \frac{1}{a_{k-1}} \leq \frac{1}{1} \\ &= 2 \end{aligned}$$

$$\begin{aligned} a_{k+1} &= \frac{1}{2}(a_k + \frac{2}{a_{k-1}}) \\ &= \frac{a_k}{2} + \frac{1}{a_{k-1}} \\ &\geq \frac{1}{2} + \frac{1}{a_{k-1}} && \text{By IH, since } a_k \geq 1 \\ &\geq \frac{1}{2} + \frac{1}{2} && \text{By IH, since } a_{k-1} \leq 2 \text{, so } \frac{1}{a_{k-1}} \geq \frac{1}{2} \\ &= 1 \end{aligned}$$

So $1 \leq a_{k+1} \leq 2$.
Conclusion: Thus we have proven P(n) for all integers $n \geq 1$ by strong induction.

4. Modular Arithmetic [10 points]
(a) Prove or disprove: If $a \equiv b \pmod{10}$, then $a \equiv b \pmod{5}$. [5 points]

Solution:
True. Suppose that $a \equiv b \pmod{10}$. Then $10 \mid (a-b)$. Then there exists some integer k such that $a-b = 10k$ for some integer k. In particular, $a - b = 5(2k)$. Then $5 \mid (a - b)$. So $a \equiv b \pmod{5}$.

(b) Prove or disprove: If $a \equiv b \pmod{10}$, then $a \equiv b \pmod{20}$. [5 points]

Solution:
False. For example, for $a = 1$ and $b = 11$. Then $a \equiv b \pmod{10}$, but $a \not\equiv b \pmod{20}$.

5. Irregularity [20 points]
Prove that the set of strings $\{0^n10^n : n \geq 0\}$ is not regular.

Solution:

$L = \{0^n10^n \ : \ n \geq 0\}$. Let D be an arbitrary DFA, and suppose for contradiction that D accepts L. Consider $S = \{0^n : n \geq 0\}$. Since S contains infinitely many strings and D has a finite number of states, two strings in S must end up in the same state. Say these strings are 0^i and 0^j for some $i, j \geq 0$ such that $i \neq j$. Append the string 10^i to both of these strings. The two resulting strings are:

$a = 0^i10^i$ Note that $a \in L$.

$b = 0^j10^i$ Note that $b \notin L$, since $i \neq j$.

Since a and b end up in the same state, but $a \in L$ and $b \notin L$, that state must be both an accept and reject state, which is a contradiction. Since D was arbitrary, there is no DFA that recognizes L, so L is not regular.