CSE 390Z: Mathematics for Computation Workshop

Practice 311 Midterm Solutions

Name: ____________________________________

UW ID: ____________________________________

Instructions:

- This is a **simulated practice midterm**. You will **not** be graded on your performance on this exam.
- Nevertheless, please treat this as if it is a real exam. That means that you may not discuss with your neighbors, reference outside material, or use your devices during the next 50 minute period.
- If you get stuck on a problem, consider moving on and coming back later. In the actual exam, there will likely be opportunity for partial credit.
- There are 5 problems on this exam, totaling 90 points.

1. Predicate Translation [15 points]
Let the domain of discourse be novels, comic books, movies, and TV shows. Translate the following statements to predicate logic, using the following predicates:

Novel(x) := x is a novel
Comic(x) := x is a comic book
Movie(x) := x is a movie
Show(x) := x is a TV show
Adaptation(x, y) := x is an adaptation of y

(a) (5 points) A novel cannot be adapted into both a movie and a TV show.

Solution:

$\forall x(\text{Novel}(x) \rightarrow \forall m \forall s((\text{Movie}(m) \land \text{Show}(s)) \rightarrow \neg(\text{Adaptation}(m, x) \land \text{Adaptation}(s, x)))$

(b) (5 points) Every movie is an adaptation of a novel or a comic book.

Solution:

$\forall m(\text{Movie}(m) \rightarrow \exists x(\text{Adaptation}(m, x) \land (\text{Novel}(x) \lor \text{Comic}(x))))$

(c) (5 points) Every novel has been adapted into exactly one movie.

Solution:

$\forall x(\text{Novel}(x) \rightarrow \exists m(\text{Movie}(m) \land \text{Adaptation}(m, x) \land \forall n((\text{Movie}(n) \land (n \neq m)) \rightarrow \neg\text{Adaptation}(n, x))))$

OR

$\forall x(\text{Novel}(x) \rightarrow \exists m(\text{Movie}(m) \land \text{Adaptation}(m, x) \land \forall n(\text{Adaptation}(n, x) \rightarrow (\neg\text{Movie}(n) \lor n = m))))$

OR

$\forall x(\text{Novel}(x) \rightarrow \exists m(\text{Movie}(m) \land \text{Adaptation}(m, x) \land \forall n((\text{Adaptation}(n, x) \land \text{Movie}(n)) \rightarrow (n = m))))$

*Note that a great exercise is to show that the above 3 solutions are all logically equivalent :)

2. Circuits [15 points]

The boolean function f takes in three boolean inputs x_1, x_2, x_3, and outputs $\neg((x_1 \oplus x_2) \wedge x_3)$.

Note: You may write your solutions using boolean algebra or propositional logic notation.

(a) (5 points) Draw a truth table for f.

Solution:

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	T

(b) (5 points) Write f as a sum-of-products expression.

Solution:

$(x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge \neg x_3) \vee (x_1 \wedge \neg x_2 \wedge \neg x_3) \vee (\neg x_1 \wedge x_2 \wedge \neg x_3) \vee (\neg x_1 \wedge \neg x_2 \wedge x_3) \vee (\neg x_1 \wedge \neg x_2 \wedge \neg x_3)$

(c) (5 points) Write f as a products-of-sums expression.

Solution:

$(\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee \neg x_3)$

3. Number Theory Proof [20 points]
Recall this definition of odd: $\text{Odd}(x) := \exists y(x = 2y + 1)$. Write an English proof to show that for all odd integers k, the statement $8 \mid k^2 - 1$ holds.

Hint: At some point in your proof, you'll need to show that for any integer a, $a(a + 1)$ is even. When you reach this point, feel free to break your proof up into the case where a is even, and the case where a is odd.

Solution:
Let k be an arbitrary odd integer. Then $k = 2a + 1$ for some integer a. Then $k^2 - 1 = (2a + 1)^2 - 1 = 4a^2 + 4a + 1 - 1 = 4a^2 + 4a = 4a(a + 1)$.

Consider the case where a is odd. Then $a = 2b + 1$ for some integer b. Then $k^2 - 1 = 4a(a + 1) = 4(2b + 1)(2b + 2) = 8(2b + 1)(b + 1)$. By closure of integers under multiplication and addition, $k^2 - 1 = 8c$ for an integer c. Thus in this case, $8 \mid k^2 - 1$.

Consider the case where a is even. Then $a = 2b$ for some integer b. Then $k^2 - 1 = 4a(a + 1) = 4(2b)(2b + 1) = 8b(2b + 1)$. By closure of integers under multiplication and addition, $k^2 - 1 = 8c$ for an integer c. Thus in this case, $8 \mid k^2 - 1$.

So in all cases, $8 \mid k^2 - 1$. Since k was an arbitrary odd integer, we have proved the claim.

4. Set Proof [20 points]

Suppose that for sets A, B, C, the facts $A \subseteq B$ and $B \subseteq C$ are given. Write an English proof to show that $B \times A \subseteq C \times C$.

Solution:

Suppose that for sets A, B, C, we have $A \subseteq B$ and $B \subseteq C$ (these are our givens). Let $x \in B \times A$ be arbitrary. Then by definition of Cartesian Product, $x = (y, z)$ for $y \in B$ and $z \in A$. Then since $y \in B$ and $B \subseteq C$, $y \in C$. Similarly since $z \in A$ and $A \subseteq B$, $z \in B$. Then since $z \in B$ and $B \subseteq C$, we have $z \in C$. Therefore we have shown that $y \in C$ and $z \in C$. Then by definition of Cartesian Product, $x \in C \times C$. Since x was arbitrary, we have shown $B \times A \subseteq C \times C$.

5. Induction [20 points]
Prove by induction that $(1+\pi)^n > 1 + n\pi$ for all integers $n \geq 2$.

Solution:

1. Let P(n) be the statement "$(1+\pi)^n > 1+n\pi$". We prove P(n) for all integers $n \geq 2$ by induction.

2. Base Case: When $n = 2$, the LHS is $(1+\pi)^2 = 1 + 2\pi + \pi^2$. The RHS is $1 + 2\pi$. Since $\pi^2 > 0$, $1 + 2\pi + \pi^2 > 1 + 2\pi$, so the Base Case holds.

3. Inductive Hypothesis: Suppose that P(k) holds for some arbitrary integer $k \geq 2$. Then $(1+\pi)^k > 1 + k\pi$.

4. Inductive Step:

Goal: Show $P(k+1)$, i.e. show $(1+\pi)^{k+1} > 1 + (k+1)\pi$

$$
\begin{aligned}
(1+\pi)^{k+1} &= (1+\pi)(1+\pi)^k && \text{Definition of Exponent} \\
&> (1+\pi)(1+k\pi) && \text{By IH} \\
&= 1 + \pi + k\pi + k\pi^2 && \text{Algebra} \\
&= 1 + (k+1)\pi + k\pi^2 && \text{Algebra} \\
&> 1 + (k+1)\pi && \text{Since } k\pi^2 > 0
\end{aligned}
$$

Thus $(1+\pi)^{k+1} > 1 + (k+1)\pi$. So P($k+1$) holds.

5. Thus we have proven P(n) for all integers $n \geq 2$ by induction.