CSE 390Z: Mathematics for Computation Workshop

Week 5 Workshop Solutions

0. Induction: Divides

Prove that $9 \mid (n^3 + (n+1)^3 + (n+2)^3)$ for all $n > 1$ by induction.

Solution:

Let $P(n)$ be "$9 \mid n^3 + (n+1)^3 + (n+2)^3$". We will prove $P(n)$ for all integers $n > 1$ by induction.

Base Case $(n = 2)$**:** $2^3 + (2+1)^3 + (2+2)^3 = 8 + 27 + 64 = 99 = 9 \cdot 11$, so $9 \mid 2^3 + (2+1)^3 + (2+2)^3$, so $P(2)$ holds.

Inductive Hypothesis: Assume that $9 \mid k^3 + (k+1)^3 + (k+2)^3$ for an arbitrary integer $k > 1$. Note that this is equivalent to assuming that $k^3 + (k+1)^3 + (k+2)^3 = 9j$ for some integer j by the definition of divides.

Inductive Step: Goal: Show $9 \mid (k+1)^3 + (k+2)^3 + (k+3)^3$

$$\begin{aligned}
(k+1)^3 + (k+2)^3 + (k+3)^3 &= (k^2 + 6k + 9)(k+3) + (k+1)^3 + (k+2)^3 && \text{[expanding trinomial]} \\
&= (k^3 + 6k^2 + 9k + 3k^2 + 18k + 27) + (k+1)^3 + (k+2)^3 && \text{[expanding binomial]} \\
&= 9k^2 + 27k + 27 + k^3 + (k+1)^3 + (k+2)^3 && \text{[adding like terms]} \\
&= 9k^2 + 27k + 27 + 9j && \text{[by I.H.]} \\
&= 9(k^2 + 3k + 3 + j) && \text{[factoring out 9]}
\end{aligned}$$

Since k and j are integers, $k^2 + 3k + 3 + j$ is also an integer. Therefore, by the definition of divides, $9 \mid (k+1)^3 + (k+2)^3 + (k+3)^3$, so $P(k) \rightarrow P(k+1)$ for an arbitrary integer $k > 1$.

Conclusion: $P(n)$ holds for all integers $n > 1$ by induction.

1. Induction: Equality

For any $n \in \mathbb{N}$, define S_n to be the sum of the squares of the first n positive integers, or

$$S_n = 1^2 + 2^2 + \cdots + n^2.$$

Prove that for all $n \in \mathbb{N}$, $S_n = \frac{1}{6}n(n+1)(2n+1)$.

Solution:

Let $\mathsf{P}(n)$ be the statement "$S_n = \frac{1}{6}n(n+1)(2n+1)$" defined for all $n \in \mathbb{N}$. We prove that $\mathsf{P}(n)$ is true for all $n \in \mathbb{N}$ by induction on n.

Base Case: When $n = 0$, we know the sum of the squares of the first n positive integers is the sum of no terms, so we have a sum of 0. Thus, $S_0 = 0$. Since $\frac{1}{6}(0)(0+1)((2)(0)+1) = 0$, we know that $\mathsf{P}(0)$ is true.

Inductive Hypothesis: Suppose that $\mathsf{P}(k)$ is true for some arbitrary $k \in \mathbb{N}$.

Inductive Step:

Goal: Show $P(k+1)$, i.e. show $S_{k+1} = \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$

Examining S_{k+1}, we see that

$$S_{k+1} = 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = S_k + (k+1)^2.$$

By the inductive hypothesis, we know that $S_k = \frac{1}{6}k(k+1)(2k+1)$. Therefore, we can substitute and rewrite the expression as follows:

$$\begin{aligned}
S_{k+1} &= S_k + (k+1)^2 \\
&= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\
&= (k+1)\left(\frac{1}{6}k(2k+1) + (k+1)\right) \\
&= \frac{1}{6}(k+1)\left(k(2k+1) + 6(k+1)\right) \\
&= \frac{1}{6}(k+1)\left(2k^2 + 7k + 6\right) \\
&= \frac{1}{6}(k+1)(k+2)(2k+3) \\
&= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)
\end{aligned}$$

Thus, we can conclude that $\mathsf{P}(k+1)$ is true.

Conclusion: $P(n)$ holds for all integers $n \geq 0$ by the principle of induction.

2. Induction: Inequality

Prove by induction on n that for all integers $n \geq 0$ the inequality $(3+\pi)^n \geq 3^n + n\pi 3^{n-1}$ is true.

Solution:

Let $P(n)$ be "$(3+\pi)^n \geq 3^n + n\pi 3^{n-1}$". We will prove $P(n)$ is true for all $n \in \mathbb{N}$, by induction.

Base Case: (n = 0): $(3+\pi)^0 = 1$ and $3^0 + 0 \cdot \pi \cdot 3^{-1} = 1$, since $1 \geq 1$, $P(0)$ is true.

Inductive Hypothesis: Suppose that $P(k)$ is true for some arbitrary integer $k \in \mathbb{N}$.

Inductive Step:

Goal: Show $P(k+1)$, i.e. show $(3+\pi)^{k+1} \geq 3^{k+1} + (k+1)\pi 3^{(k+1)-1} = 3^{k+1} + (k+1)\pi 3^k$

$$
\begin{aligned}
(3+\pi)^{k+1} &= (3+\pi)^k \cdot (3+\pi) && \text{(Factor out } (3+\pi)\text{)} \\
&\geq (3^k + k3^{k-1}\pi) \cdot (3+\pi) && \text{(By I.H., } (3+\pi) \geq 0\text{)} \\
&= 3 \cdot 3^k + 3^k\pi + 3k3^{k-1}\pi + k3^{k-1}\pi^2 && \text{(Distributive property)} \\
&= 3^{k+1} + 3^k\pi + k3^k\pi + k3^{k-1}\pi^2 && \text{(Simplify)} \\
&= 3^{k+1} + (k+1)3^k\pi + k3^{k-1}\pi^2 && \text{(Factor out } (k+1)\text{)} \\
&\geq 3^{k+1} + (k+1)\pi 3^k && (k3^{k-1}\pi^2 \geq 0)
\end{aligned}
$$

Conclusion: So by induction, $P(n)$ is true for all $n \in \mathbb{N}$.

3. Induction: Another Inequality

Prove by induction on n that for all integers $n \geq 4$ the inequality $n! > 2^n$ is true.

Solution:

Let $P(n)$ be "$n! > 2^n$". We will prove $P(n)$ is true for all $n \in \mathbb{N}$, $n \geq 4$, by induction.

Base Case: (n = 4)**:** $4! = 24$ and $2^4 = 16$, since $24 > 16$, $P(4)$ is true.

Inductive Hypothesis: Suppose that $P(k)$ is true for some arbitrary integer $k \in \mathbb{N}$, $k \geq 4$.

Inductive Step:

Goal: Show $P(k+1)$, i.e. show $(k+1)! > 2^{k+1}$

$$
\begin{aligned}
(k+1)! &= k! \cdot (k+1) \\
&> 2^k \cdot (k+1) && \text{(By I.H., } k! > 2^k\text{)} \\
&> 2^k \cdot 2 && \text{(Since } k \geq 4\text{, so } k+1 \geq 5 > 2\text{)} \\
&= 2^{k+1}
\end{aligned}
$$

Conclusion: So by induction, $P(n)$ is true for all $n \in \mathbb{N}$, $n \geq 4$.

4. Strong Induction: Stamp Collection

A store sells 3 cent and 5 cent stamps. Use strong induction to prove that you can make exactly n cents worth of stamps for all $n \geq 10$.

Hint: you'll need multiple base cases for this - think about how many steps back you need to go for your inductive step.

Solution:

Let P(n) be defined as "You can buy exactly n cents of stamps". We will prove $P(n)$ is true for all integers $n \geq 10$ by strong induction.

Base Cases: $(n = 10, 11, 12)$**:**

- $n = 10$: 10 cents of stamps can be made from two 5 cent stamps.
- $n = 11$: 11 cents of stamps can be made from one 5 cent and two 3 cent stamps.
- $n = 12$: 12 cents of stamps can be made from four 3 cent stamps.

Inductive Hypothesis: Suppose for some arbitrary integer $k \geq 12$, P(10) $\wedge$P(11) $\wedge$... $\wedge$P(k) holds.

Inductive Step:

Goal: Show $P(k+1)$, i.e. show that we can make $k+1$ cents in stamps.

We want to buy $k+1$ cents in stamps. By the I.H., we can buy exactly $(k+1)-3 = k-2$ cents in stamps. Then, we can add another 3 cent stamp in order to buy $k+1$ cents in stamps, so P$(k+1)$ is true.

Note: How did we decide how many base cases to have? Well, we wanted to be able to assume P$(k-2)$, and add 3 to achieve P$(k+1)$. Therefore we needed to be able to assume that $k-2 \geq 10$. Adding 2 to both sides, we needed to be able to assume that $k \geq 12$. So, we have to prove the base cases up to 12, that is: $10, 11, 12$.

Another way to think about this is that we had to use a fact from 3 steps back from $k+1$ to $k-2$ in the IS, so we needed 3 base cases.

Conclusion: So by strong induction, P(n) is true for all integers $n \geq 10$.

5. Strong Induction: Functions

Consider the function $f(n)$ defined for integers $n \geq 1$ as follows:
$f(1) = 1$ for $n = 1$
$f(2) = 4$ for $n = 2$
$f(3) = 9$ for $n = 3$
$f(n) = f(n-1) - f(n-2) + f(n-3) + 2(2n-3)$ for $n \geq 4$

Prove by strong induction that for all $n \geq 1$, $f(n) = n^2$.

Solution:

Let P(n) be defined as " $f(n) = n^2$". We will prove $P(n)$ is true for all integers $n \geq 1$ by strong induction.

Base Cases: $(n = 1, 2, 3)$**:**

- $n = 1$: $f(1) = 1 = 1^2$.
- $n = 2$: $f(2) = 4 = 2^2$.
- $n = 3$: $f(3) = 9 = 3^2$

So the base cases hold.

Inductive Hypothesis: Suppose for some arbitrary integer $k \geq 3$, P$(1) \wedge \ldots \wedge$P(k) hold.

Inductive Step:

Goal: Show $P(k+1)$, i.e. show that $f(k+1) = (k+1)^2$.

$$\begin{aligned}
f(k+1) &= f(k+1-1) - f(k+1-2) + f(k+1-3) + 2(2(k+1)-3) && \text{Definition of f} \\
&= f(k) - f(k-1) + f(k-2) + 2(2k-1) \\
&= k^2 - (k-1)^2 + (k-2)^2 + 2(2k-1) && \text{By IH} \\
&= k^2 - (k^2 - 2k + 1) + (k^2 - 4k + 4) + 4k - 2 \\
&= (k^2 - k^2 + k^2) + (2k - 4k + 4k) + (-1 + 4 - 2) \\
&= k^2 + 2k + 1 \\
&= (k+1)^2
\end{aligned}$$

So P$(k+1)$ holds.

Conclusion: So by strong induction, P(n) is true for all integers $n \geq 1$.

6. Strong Induction: Collecting Candy

A store sells candy in packs of 4 and packs of 7. Let P(n) be defined as "You are able to buy n packs of candy". For example, $P(3)$ is not true, because you cannot buy exactly 3 packs of candy from the store. However, it turns out that P(n) is true for any $n \geq 18$. Use strong induction on n to prove this.

Hint: you'll need multiple base cases for this - think about how many steps back you need to go for your inductive step.

Solution:

Let P(n) be defined as "You are able to buy n packs of candy". We will prove $P(n)$ is true for all integers $n \geq 18$ by strong induction.

Base Cases: $(n = 18, 19, 20, 21)$**:**

- $n = 18$: 18 packs of candy can be made up of 2 packs of 7 and 1 pack of 4 $(18 = 2 * 7 + 1 * 4)$.
- $n = 19$: 19 packs of candy can be made up of 1 pack of 7 and 3 packs of 4 $(19 = 1 * 7 + 3 * 4)$.
- $n = 20$: 20 packs of candy can be made up of 5 packs of 4 $(20 = 5 * 4)$.
- $n = 21$: 21 packs of candy can be made up of 3 packs of 7 $(21 = 3 * 7)$.

Inductive Hypothesis: Suppose for some arbitrary integer $k \geq 21$, P(18) $\wedge$... $\wedge$P(k) hold.

Inductive Step:

Goal: Show $P(k + 1)$, i.e. show that we can buy $k + 1$ packs of candy.

We want to buy $k+1$ packs of candy. By the I.H., we can buy exactly $k-3$ packs, so we can add another pack of 4 packs in order to buy $k + 1$ packs of candy, so P$(k + 1)$ is true.

Note: How did we decide how many base cases to have? Well, we wanted to be able to assume P$(k-3)$, and add 4 to achieve P$(k + 1)$. Therefore we needed to be able to assume that $k - 3 \geq 18$. Adding 3 to both sides, we needed to be able to assume that $k \geq 21$. So, we have to prove the base cases up to 21, that is: $18, 19, 20, 21$.

Another way to think about this is that we had to use a fact from 4 steps back from $k + 1$ to $k - 3$ in the IS, so we needed 4 base cases.

Conclusion: So by strong induction, P(n) is true for all integers $n \geq 18$.