CSE 390Z: Mathematics for Computation Workshop

Week 5 Workshop

Conceptual Review

(a) **Set Definitions**

Set Equality: $A = B := \forall x(x \in A \leftrightarrow x \in B)$
Subset: $A \subseteq B := \forall x(x \in A \rightarrow x \in B)$
Union: $A \cup B := \{x \ : \ x \in A \vee x \in B\}$
Intersection: $A \cap B := \{x \ : \ x \in A \wedge x \in B\}$
Set Difference: $A \setminus B = A - B := \{x \ : \ x \in A \wedge x \notin B\}$
Set Complement: $\overline{A} = A^C := \{x \ : \ x \notin A\}$
Powerset: $\mathcal{P}(A) := \{B \ : B \subseteq A\}$
Cartesian Product: $A \times B := \{(a, b) \ : a \in A, \ b \in B\}$

(b) How do we prove that for sets A and B, $A \subseteq B$?

Solution:

Let $x \in A$ be arbitrary... thus $x \in B$. Since x was arbitrary, $A \subseteq B$.

(c) How do we prove that for sets A and B, $A = B$?

Solution:

Method 1: Use two subset proofs to show that $A \subseteq B$ and $B \subseteq A$.
Method 2: Use a chain of logical equivalences.

(d) What does $\{x \in \mathbb{Z} \ : \ x > 0\}$ mean? **Note:** this notation is called "set-builder" notation.

Solution:

The set of all positive integers.

1. Examples

(a) Prove that $A \cap B \subseteq A \cup B$.

Solution:

Let $x \in A \cap B$ be arbitrary. Then by definition of intersection, $x \in A$ and $x \in B$. So certainly $x \in A$ or $x \in B$. Then by definition of union, $x \in A \cup B$.

(b) Prove that $A \cap (A \cup B) = A \cup (A \cap B)$ with a chain of equivalences proof.

Solution:

Let x be arbitrary. Observe that:

$$\begin{aligned} x \in A \cap (A \cup B) &\equiv (x \in A) \wedge (x \in A \cup B) && \text{Def of Intersection} \\ &\equiv (x \in A) \wedge ((x \in A) \vee (x \in B)) && \text{Def of Union} \\ &\equiv ((x \in A) \wedge (x \in A)) \vee ((x \in A) \wedge (x \in B)) && \text{Distributivity} \\ &\equiv (x \in A) \vee ((x \in A) \wedge (x \in B)) && \text{Idempotency} \end{aligned}$$

$\equiv (x \in A) \vee (x \in A \cap B)$ Def of Intersection

$\equiv x \in A \cup (A \cap B)$ Def of Union

Since x was arbitrary, we have shown $A \cap (A \cup B) = A \cup (A \cap B)$.

2. Set Operations

Let $A = \{1, 2, 5, 6, 8\}$ and $B = \{2, 3, 5\}$.

(a) What is the set $A \cap (B \cup \{2, 8\})$?

Solution:

$\{2, 5, 8\}$

(b) What is the set $\{10\} \cup (A \setminus B)$?

Solution:

$\{1, 6, 8, 10\}$

(c) What is the set $\mathcal{P}(B)$?

Solution:

$\{\{2, 3, 5\}, \{2, 3\}, \{2, 5\}, \{3, 5\}, \{2\}, \{3\}, \{5\}, \emptyset\}$

(d) How many elements are in the set $A \times B$? List 3 of the elements.

Solution:

15 elements, for example $(1, 2), (1, 3), (1, 5)$.

3. Set Equality Proof

(a) Write an English proof to show that $A \cap (A \cup B) \subseteq A$ for any sets A, B.

Solution:

Let x be an arbitrary member of $A \cap (A \cup B)$. Then by definition of intersection, $x \in A$ and $x \in A \cup B$. So certainly, $x \in A$. Since x was arbitrary, $A \cap (A \cup B) \subseteq A$.

(b) Write an English proof to show that $A \subseteq A \cap (A \cup B)$ for any sets A, B.

Solution:

Let $y \in A$ be arbitrary. So certainly $y \in A$ or $y \in B$. Then by definition of union, $y \in A \cup B$. Since $y \in A$ and $y \in A \cup B$, by definition of intersection, $y \in A \cap (A \cup B)$. Since y was arbitrary, $A \subseteq A \cap (A \cup B)$.

(c) Combine part (a) and (b) to conclude that $A \cap (A \cup B) = A$ for any sets A, B.

Solution:

Since $A \cap (A \cup B) \subseteq A$ and $A \subseteq A \cap (A \cup B)$, we can deduce that $A \cap (A \cup B) = A$.

(d) Prove $A \cap (A \cup B) = A$ again, but using a **chain of equivalences proof** instead.

Solution:

Let x be arbitrary. Observe:

$$\begin{aligned} x \in A \cap (A \cup B) &\equiv (x \in A) \wedge (x \in A \cup B) && \text{Def of Intersection} \\ &\equiv (x \in A) \wedge ((x \in A) \vee (x \in B)) && \text{Def of Union} \\ &\equiv x \in A && \text{Absorption} \end{aligned}$$

Since x was arbitrary, we have shown $A \cap (A \cup B) = A$.

4. Subsets

Prove or disprove: for any sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Solution:

Let A, B, C be sets, and suppose $A \subseteq B$ and $B \subseteq C$. Let x be an arbitrary element of A. Then, by definition of subset, $x \in B$, and by definition of subset again, $x \in C$. Since x was an arbitrary element of A, we see that all elements of A are in C, so by definition of subset, $A \subseteq C$. So, for any sets A, B, C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

5. $\cup \to \cap$?

Prove or disprove: for all sets A and B, $A \cup B \subseteq A \cap B$.

Solution:

We wish to disprove this claim via a counterexample. Choose $A = \{1\}$, $B = \varnothing$. Note that $A \cup B = \{1\} \cup \varnothing = \{1\}$ by definition of set union. Note that $A \cap B = \{1\} \cap \varnothing = \varnothing$ by definition of set intersection. $\{1\} \not\subseteq \varnothing$, so the claim does not hold for these sets. Since we found a counterexample to the claim, we have shown that it is not the case that $A \cup B \not\subseteq A \cap B$ for all sets A and B.

6. Cartesian Product Proof

Write an English proof to show that $A \times C \subseteq (A \cup B) \times (C \cup D)$.

Solution:

Let $x \in A \times C$ be arbitrary. Then x is of the form $x = (y, z)$, where $y \in A$ and $z \in C$. Then certainly $y \in A$ or $y \in B$. Then by definition of union, $y \in (A \cup B)$. Similarly, since $z \in C$, certainly $z \in C$ or $z \in D$. Then by definition, $z \in (C \cup D)$. Since $x = (y, z)$, then $x \in (A \cup B) \times (C \cup D)$. Since x was arbitrary, we have shown $A \times C \subseteq (A \cup B) \times (C \cup D)$.

7. Set Equality Proof

We want to prove that $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

(a) First prove this with a chain of logical equivalences proof.

Solution:

Let x be arbitrary. Observe:

$$\begin{aligned} x \in A \setminus (B \cap C) &\equiv (x \in A) \wedge (x \notin B \cap C) && \text{Def of Set Difference} \\ &\equiv (x \in A) \wedge \neg(x \in B \cap C) && \text{Def of element} \\ &\equiv (x \in A) \wedge \neg((x \in B) \wedge (x \in C)) && \text{Def of Intersection} \\ &\equiv (x \in A) \wedge (\neg(x \in B) \vee \neg(x \in C)) && \text{DeMorgan's Law} \\ &\equiv (x \in A) \wedge ((x \notin B) \vee (x \notin C)) && \text{Def of element} \end{aligned}$$

$$\equiv ((x \in A) \wedge (x \notin B)) \vee ((x \in A) \wedge (x \notin C)) \qquad \text{Distributivity}$$
$$\equiv (x \in A \setminus B) \vee (x \in A \setminus C) \qquad \text{Def of Set Difference}$$
$$\equiv x \in (A \setminus B) \cup (A \setminus C) \qquad \text{Def of Union}$$

Since x was arbitrary, we have shown $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

(b) Now prove this with an English proof that is made of two subset proofs.

Solution:

Let $x \in A \setminus (B \cap C)$ be arbitrary. Then by definition of set difference, $x \in A$ and $x \notin B \cap C$. Then by definition of intersection, $x \notin B$ or $x \notin C$. Thus (by distributive property of propositions) we have $x \in A$ and $x \notin B$, or $x \in A$ and $x \notin C$. Then by definition of set difference, $x \in (A \setminus B)$ or $x \in (A \setminus C)$. Then by definition of union, $x \in (A \setminus B) \cup (A \setminus C)$. Since x was arbitrary, we have shown $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$.

Let $x \in (A \setminus B) \cup (A \setminus C)$ be arbitrary. Then by definition of union, $x \in (A \setminus B)$ or $x \in (A \setminus C)$. Then by definition of set difference, $x \in A$ and $x \notin B$, or $x \in A$ and $x \notin C$. Then (by distributive property of propositions) $x \in A$, and $x \notin B$ or $x \notin C$. Then by definition of intersection, $x \in A$ and $x \notin (B \cap C)$. Then by definition of set difference, $x \in A \setminus (B \cap C)$. Since x was arbitrary, we have shown that $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$.

Since $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$ and $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$, we have shown $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

8. Constructing Sets

Use set builder notation to construct the following sets. You may use arithmetic predicates $=, <, >, \leq, \geq, \neq$, and arithmetic operations $+, \cdot, -, \div$.

Recall that integers are the numbers $\{... -2, -1, 0, 1, 2...\}$, and are denote $\mathbb{Z}$.

(a) The set of even integers.

Solution:

$\{2x \ : \ x \in \mathbb{Z}\}$ or $\{x \ : \ x = 2k, k \in \mathbb{Z}\}$ or $\{x \in \mathbb{Z} \ : 2|x\}$

(b) The set of integers that are one more than a perfect square.

Solution:

$\{x^2 + 1 \ : \ x \in \mathbb{Z}\}$

(c) The set of integers that are greater than 5.

Solution:

$\{x \in \mathbb{Z} \ : x > 5\}$

9. Making a Difference

Garrett and Shaoqi are working on their AI homework and tell you the following. Let G denote the set of AI homework questions that Garrett has not yet solved. Let S denote the set of AI homework questions that Shaoqi has not yet solved. Garrett and Shaoqi claim that $G \setminus S = S \setminus G$.

In what circumstance is this true? In what circumstance is it false? Can you justify this (formal proof not required)?

Solution:

This is only true in the case when $G = S$. In all other cases, $G \setminus S \neq S \setminus G$.

Justification:
When $G = S$, $G \setminus S = \emptyset$ and $S \setminus G = \emptyset$. So $G \setminus S \neq S \setminus G$ holds.

When $G \neq S$, then either there exists some element x such that $x \in G$ and $x \notin S$, or some element y such that $y \in S$ and $y \notin G$. Assume we are in the first case (the second case follows a similar argument). Then because $x \in G$ and $x \notin S$, x will be in $G \setminus S$. However, since $x \notin S$, x will not be in $S \setminus G$. Thus in this case, $G \setminus S \neq S \setminus G$.