Week 5 Workshop

Conceptual Review

(a) Set Definitions

Set Equality: \( A = B := \forall x (x \in A \leftrightarrow x \in B) \)

Subset: \( A \subseteq B := \forall x (x \in A \rightarrow x \in B) \)

Union: \( A \cup B := \{x : x \in A \lor x \in B\} \)

Intersection: \( A \cap B := \{x : x \in A \land x \in B\} \)

Set Difference: \( A \setminus B = A - B := \{x : x \in A \land x \notin B\} \)

Set Complement: \( A^c := \{x : x \notin A\} \)

Powerset: \( \mathcal{P}(A) := \{B : B \subseteq A\} \)

Cartesian Product: \( A \times B := \{(a, b) : a \in A, b \in B\} \)

(b) How do we prove that for sets \( A \) and \( B \), \( A \subseteq B \)?

Solution:
Let \( x \in A \) be arbitrary... thus \( x \in B \). Since \( x \) was arbitrary, \( A \subseteq B \).

(c) How do we prove that for sets \( A \) and \( B \), \( A = B \)?

Solution:
Method 1: Use two subset proofs to show that \( A \subseteq B \) and \( B \subseteq A \).
Method 2: Use a chain of logical equivalences.

(d) What does \( \{x \in \mathbb{Z} : x > 0\} \) mean? Note: this notation is called "set-builder" notation.

Solution:
The set of all positive integers.

1. Examples

(a) Prove that \( A \cap B \subseteq A \cup B \).

Solution:
Let \( x \in A \cap B \) be arbitrary. Then by definition of intersection, \( x \in A \) and \( x \in B \). So certainly \( x \in A \) or \( x \in B \). Then by definition of union, \( x \in A \cup B \).

(b) Prove that \( A \cap (A \cup B) = A \cup (A \cap B) \) with a chain of equivalences proof.

Solution:
Let \( x \) be arbitrary. Observe that:

\[
\begin{align*}
x \in A \cap (A \cup B) \equiv (x \in A) \land (x \in A \cup B) \\
\equiv (x \in A) \land ((x \in A) \lor (x \in B)) \\
\equiv ((x \in A) \land (x \in A)) \lor ((x \in A) \land (x \in B)) \\
\equiv (x \in A) \lor ((x \in A) \land (x \in B)) \\
\end{align*}
\]

Def of Intersection
Def of Union
Distributivity
Idempotency
\( \equiv (x \in A) \lor (x \in A \cap B) \)  
\( \equiv x \in A \cup (A \cap B) \)  

Def of Intersection  
Def of Union

Since \( x \) was arbitrary, we have shown \( A \cap (A \cup B) = A \cup (A \cap B) \).

2. Set Operations

Let \( A = \{1, 2, 5, 6, 8\} \) and \( B = \{2, 3, 5\} \).

(a) What is the set \( A \cap (B \cup \{2, 8\}) \)?

Solution:
\{2, 5, 8\}

(b) What is the set \( \{10\} \cup (A \setminus B) \)?

Solution:
\{1, 6, 8, 10\}

(c) What is the set \( P(B) \)?

Solution:
\{\{2, 3, 5\}, \{2, 3\}, \{2, 5\}, \{3\}, \{5\}, \emptyset\}

(d) How many elements are in the set \( A \times B \)? List 3 of the elements.

Solution:
15 elements, for example \((1, 2), (1, 3), (1, 5)\).

3. Set Equality Proof

(a) Write an English proof to show that \( A \cap (A \cup B) \subseteq A \) for any sets \( A, B \).

Solution:
Let \( x \) be an arbitrary member of \( A \cap (A \cup B) \). Then by definition of intersection, \( x \in A \) and \( x \in A \cup B \). So certainly, \( x \in A \). Since \( x \) was arbitrary, \( A \cap (A \cup B) \subseteq A \).

(b) Write an English proof to show that \( A \subseteq A \cap (A \cup B) \) for any sets \( A, B \).

Solution:
Let \( y \in A \) be arbitrary. So certainly \( y \in A \) or \( y \in B \). Then by definition of union, \( y \in A \cup B \). Since \( y \in A \) and \( y \in A \cup B \), by definition of intersection, \( y \in A \cap (A \cup B) \). Since \( y \) was arbitrary, \( A \subseteq A \cap (A \cup B) \).

(c) Combine part (a) and (b) to conclude that \( A \cap (A \cup B) = A \) for any sets \( A, B \).

Solution:
Since \( A \cap (A \cup B) \subseteq A \) and \( A \subseteq A \cap (A \cup B) \), we can deduce that \( A \cap (A \cup B) = A \).

(d) Prove \( A \cap (A \cup B) = A \) again, but using a chain of equivalences proof instead.
We want to prove that

4. Subsets

Prove or disprove: for any sets $A$, $B$, and $C$, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Solution:

Let $A$, $B$, $C$ be sets, and suppose $A \subseteq B$ and $B \subseteq C$. Let $x$ be an arbitrary element of $A$. Then, by definition of subset, $x \in B$, and by definition of subset again, $x \in C$. Since $x$ was an arbitrary element of $A$, we see that all elements of $A$ are in $C$, so by definition of subset, $A \subseteq C$. So, for any sets $A$, $B$, $C$, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

5. $\cup \rightarrow \cap$?

Prove or disprove: for all sets $A$ and $B$, $A \cup B \subseteq A \cap B$.

Solution:

We wish to disprove this claim via a counterexample. Choose $A = \{1\}$, $B = \emptyset$. Note that $A \cup B = \{1\} \cup \emptyset = \{1\}$ by definition of set union. Note that $A \cap B = \{1\} \cap \emptyset = \emptyset$ by definition of set intersection. $\{1\} \not\subseteq \emptyset$, so the claim does not hold for these sets. Since we found a counterexample to the claim, we have shown that it is not the case that $A \cup B \not\subseteq A \cap B$ for all sets $A$ and $B$.

6. Cartesian Product Proof

Write an English proof to show that $A \times C \subseteq (A \cup B) \times (C \cup D)$.

Solution:

Let $x \in A \times C$ be arbitrary. Then $x$ is of the form $x = (y, z)$, where $y \in A$ and $z \in C$. Then certainly $y \in A$ or $y \in B$. Then by definition of union, $y \in (A \cup B)$. Similarly, since $z \in C$, certainly $z \in C$ or $z \in D$. Then by definition, $z \in (C \cup D)$. Since $x = (y, z)$, then $x \in (A \cup B) \times (C \cup D)$. Since $x$ was arbitrary, we have shown $A \times C \subseteq (A \cup B) \times (C \cup D)$.

7. Set Equality Proof

We want to prove that $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

(a) First prove this with a chain of logical equivalences proof.

Solution:

Let $x$ be arbitrary. Observe:

$$x \in A \setminus (B \cap C) \equiv (x \in A) \land (x \notin B \cap C) \equiv (x \in A) \land \neg(x \in B \cap C) \equiv (x \in A) \land \neg((x \in B) \land (x \in C)) \equiv (x \in A) \land \neg((x \in B) \lor \neg(x \in C)) \equiv (x \in A) \land ((x \notin B) \lor (x \notin C))$$

Def of Set Difference

Def of element

Def of Intersection

DeMorgan’s Law

Def of element
\[(x \in A \land x \notin B) \lor (x \in A \land x \notin C) \quad \text{Distributivity}\]
\[(x \in A \setminus B) \lor (x \in A \setminus C) \quad \text{Def of Set Difference}\]
\[x \in (A \setminus B) \cup (A \setminus C) \quad \text{Def of Union}\]

Since \(x\) was arbitrary, we have shown \(A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)\).

(b) Now prove this with an English proof that is made of two subset proofs.

**Solution:**

Let \(x \in A \setminus (B \cap C)\) be arbitrary. Then by definition of set difference, \(x \in A\) and \(x \notin B \cap C\). Then by definition of intersection, \(x \notin B\) or \(x \notin C\). Thus (by distributive property of propositions) we have \(x \in A\) and \(x \notin B\), or \(x \in A\) and \(x \notin C\). Then by definition of set difference, \(x \in (A \setminus B)\) or \(x \in (A \setminus C)\). Then by definition of union, \(x \in (A \setminus B) \cup (A \setminus C)\). Since \(x\) was arbitrary, we have shown \(A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)\).

Let \(x \in (A \setminus B) \cup (A \setminus C)\) be arbitrary. Then by definition of union, \(x \in (A \setminus B)\) or \(x \in (A \setminus C)\). Then by definition of set difference, \(x \in A\) and \(x \notin B\), or \(x \in A\) and \(x \notin C\). Then (by distributive property of propositions) \(x \in A\), and \(x \notin B\) or \(x \notin C\). Then by definition of intersection, \(x \in A\) and \(x \notin (B \cap C)\). Then by definition of set difference, \(x \in A \setminus (B \cap C)\). Since \(x\) was arbitrary, we have shown that \((A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)\).

Since \(A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)\) and \((A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)\), we have shown \(A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)\).

8. Constructing Sets

Use set builder notation to construct the following sets. You may use arithmetic predicates =, \(<\), \(>\), \(\leq\), \(\geq\), \(\neq\), and arithmetic operations +, -, \(\cdot\), \(\div\).

Recall that integers are the numbers \(\{... -2, -1, 0, 1, 2...\}\), and are denote \(\mathbb{Z}\).

(a) The set of even integers.

**Solution:**

\(\{2x : x \in \mathbb{Z}\}\) or \(\{x : x = 2k, k \in \mathbb{Z}\}\) or \(\{x \in \mathbb{Z} : 2|x\}\)

(b) The set of integers that are one more than a perfect square.

**Solution:**

\(\{x^2 + 1 : x \in \mathbb{Z}\}\)

(c) The set of integers that are greater than 5.

**Solution:**

\(\{x \in \mathbb{Z} : x > 5\}\)
9. Making a Difference
Garrett and Shaoqi are working on their AI homework and tell you the following. Let \( G \) denote the set of AI homework questions that Garrett has not yet solved. Let \( S \) denote the set of AI homework questions that Shaoqi has not yet solved. Garrett and Shaoqi claim that \( G \setminus S = S \setminus G \).

In what circumstance is this true? In what circumstance is it false? Can you justify this (formal proof not required)?

Solution:
This is only true in the case when \( G = S \). In all other cases, \( G \setminus S \neq S \setminus G \).

Justification:
When \( G = S \), \( G \setminus S = \emptyset \) and \( S \setminus G = \emptyset \). So \( G \setminus S \neq S \setminus G \) holds.

When \( G \neq S \), then either there exists some element \( x \) such that \( x \in G \) and \( x \notin S \), or some element \( y \) such that \( y \in S \) and \( y \notin G \). Assume we are in the first case (the second case follows a similar argument). Then because \( x \in G \) and \( x \notin S \), \( x \) will be in \( G \setminus S \). However, since \( x \notin S \), \( x \) will not be in \( S \setminus G \). Thus in this case, \( G \setminus S \neq S \setminus G \).