CSE 390Z: Mathematics for Computation Workshop

Week 3 Workshop Solutions

Conceptual Review

(a) What is a predicate, a domain of discourse, and a quantifier?

Solution:

Predicate: A function, usually based on one or more variables, that is true or false.

Domain of Discourse: The universe of values that variables come from.

Quantifier: A claim about when the predicate is true. There are two quantifiers. $\forall$ says that the claim is true for all values, and $\exists$ says there exists a value for which the claim is true.

(b) When translating to predicate logic, how do you restrict to a smaller domain in a "for all"? How do you restrict to a smaller domain in an "exists"?

Solution:

If we need to restrict something quantified by a "for all", we use **implication**. If we need to restrict something quantifies by an "exists", we use **and**.

For example, suppose the domain of discourse is all animals. We translate "all birds can fly" to $\forall x(\text{Bird}(x) \rightarrow \text{Fly}(x))$. We translate "there is a bird that can fly" to $\exists x(\text{Bird}(x) \wedge \text{Fly}(x))$.

(c) Inference Proof Rules:

Introduce $\vee$: $\frac{A}{\therefore\ A\vee B,\ B\vee A}$

Eliminate $\vee$: $\frac{A\vee B\ ;\ \neg A}{\therefore\ B}$

Introduce $\wedge$: $\frac{A\ ;\ B}{\therefore\ A\wedge B}$

Eliminate $\wedge$: $\frac{A\wedge B}{\therefore\ A\ ,\ B}$

Direct Proof: $\frac{A\Rightarrow B}{\therefore\ A\rightarrow B}$

Modus Ponens: $\frac{A\ ;\ A\rightarrow B}{\therefore\ B}$

1. Predicate Logic: Warmup

Let the domain of discourse be all animals. Let Cat(x) ::= "x is a cat" and Blue(x) ::= "x is blue". Translate the following statements to English.

(a) $\forall x(\text{Cat}(x) \land \text{Blue}(x))$

Solution:

All animals are blue cats.

(b) $\forall x(\text{Cat}(x) \rightarrow \text{Blue}(x))$

Solution:

All cats are blue.

(c) $\exists x(\text{Cat}(x) \land \text{Blue}(x))$

Solution:

There exists a blue cat.

(d) Kabir translated the sentence "there exists a blue cat" to $\exists x(\text{Cat}(x) \rightarrow \text{Blue}(x))$. This is wrong! Let's understand why.

Use the Law of Implications to rewrite Kabir's translation without the $\rightarrow$.

Solution:

$\exists x(\neg\text{Cat}(x) \lor \text{Blue}(x))$

(e) Translate the predicate from (d) back to English. How does this differ from the intended meaning?

Solution:

Translation: There exists an animal that is not a cat, or is blue.

The difference: If there was even one non-cat animal in the universe (e.g. a single dog), this condition would be satisfied. Similarly, if there was even one blue animal in the universe, this condition would be satisfied. So, this is a very different condition than "there exists a blue cat".

(f) This is a warning to be very careful when including an implication nested under an exists! (Nothing to write for this part).

2. Predicate Logic: Domains of Discourse

For the following, find a domain of discourse where the following statement is true and another where it is false. Note that for the arithmetic symbols to make sense, the domains of discourse should be sets of numbers.

(a) $\exists x(2x = 0)$

Solution:

True domain: Any set of numbers that includes 0; e.g. all natural numbers.
False domain: Any set of numbers that doesn't include 0; e.g. all integers greater than 0.

(b) $\forall x \exists y(x + y = 0)$

Solution:

True domain: Any set of numbers that includes additive inverses; e.g. all integers.
False domain: Any set of numbers that doesn't include additive inverses; e.g. all positive integers.

(c) $\exists x \forall y(x + y = y)$

Solution:

True domain: Any set of numbers that includes 0; e.g. all natural numbers (if $x = 0$, the statement holds for all y).
False domain: Any set of numbers that doesn't include 0; e.g. all integers greater than 0.

3. Predicate Logic: English to Logic

Express the following sentences in predicate logic. The domain of discourse is penguins. You may use the following predicates: $\text{Love}(x, y)$ ::= "x loves y", $\text{Dances}(x)$ ::= "x dances", $\text{Sings}(x)$ ::= "x sings", as well as $=$ and $\neq$.

(a) There is a penguin that every penguin loves.

Solution:

$\exists x \forall y(\text{Loves}(y, x))$

(b) All penguins that sing love a penguin that does not sing.

Solution:

$\forall x(\text{Sings}(x) \rightarrow \exists y(\neg\text{Sings}(y) \wedge \text{Loves}(x, y)))$

(c) There is exactly one penguin that dances.

Solution:

$\exists x(\text{Dances}(x) \wedge \forall y((y \neq x) \rightarrow \neg\text{Dances}(y)))$

or, equivalently:

$\exists x(\text{Dances}(x) \wedge \forall y(\text{Dances}(y) \rightarrow (x = y)))$

(d) There exists a penguin that loves itself, but hates (does not love) every other penguin.

Solution:

$\exists x(\text{Loves}(x,x) \wedge \forall y((y \neq x) \rightarrow \neg\text{Loves}(x,y)))$

4. Predicate Logic: Logic to English

Translate the following sentences to English. Assume the same predicates and domain of discourse as the previous problem.

(a) $\neg\exists x(\text{Dances}(x))$

Solution:

No penguins dance.

(b) $\exists x \forall y(\text{Loves}(x,y))$

Solution:

There is a penguin that loves all penguins.

(c) $\forall x(\text{Dances}(x) \rightarrow \exists y(\text{Loves}(y,x)))$

Solution:

All penguins that dance have a penguin that loves them.

(d) $\exists x \forall y((\text{Dances}(y) \wedge \text{Sings}(y)) \rightarrow \text{Loves}(x,y))$

Solution:

There exists a penguin that loves all penguins who dance and sing.

5. Formal Proofs: Warmup

(a) Given $p \wedge q$, prove $p \vee q$.

Solution:

1.	$p \wedge a$	(Given)
2.	p	(Elim $\wedge$: 1.)
3.	$p \vee q$	(Intro $\vee$: 2.)

(b) Given $p \rightarrow r$, $r \rightarrow s$, prove $p \rightarrow s$.

Solution:

1.	$p \rightarrow r$	(Given)
2.	$r \rightarrow s$	(Given)
3.1	p	(Assumption)
3.2	r	(Modus Ponens: 3.1, 1)
3.3	s	(Modus Ponens: 3.2, 2)
3.	$p \rightarrow s$	(Direct Proof Rule: 3.1-3.3)

6. Formal Proofs: Modus Ponens

(a) Prove that given $p \rightarrow q$, $\neg s \rightarrow \neg q$, and p, we can conclude s.

Solution:

1.	$p \rightarrow q$	(Given)
2.	$\neg s \rightarrow \neg q$	(Given)
3.	p	(Given)
4.	q	(Modus Ponens: 1,3)
5.	$q \rightarrow s$	(Contrapositive: 2)
6.	s	(Modus Ponens: 5,4)

(b) Prove that given $\neg(p \vee q) \rightarrow s$, $\neg p$, and $\neg s$, we can conclude q.

Solution:

1.	$\neg(p \vee q) \rightarrow s$	(Given)
2.	$\neg p$	(Given)
3.	$\neg s$	(Given)
4.	$\neg s \rightarrow \neg\neg(p \vee q)$	(Contrapositive: 1)
5.	$\neg s \rightarrow (p \vee q)$	(Double Negation: 4)
6.	$p \vee q$	(Modus Ponens: 3,5)
7.	q	(Elim $\vee$: 6,2)

7. Formal Proofs: Direct Proof Rule

(a) Prove that given $p \rightarrow q$, we can conclude $(p \wedge r) \rightarrow q$

Solution:

1.	$p \rightarrow q$	(Given)
2.1	$p \wedge r$	(Assumption)
2.2	p	(Elim $\wedge$: 2.1)
2.3	q	(Modus Ponens: 2.2, 1)
2.	$(p \wedge r) \rightarrow q$	(Direct Proof Rule: 2.1-2.3)

(b) Prove that given $p \vee q$, $q \rightarrow r$, and $r \rightarrow s$, we can conclude $\neg p \rightarrow s$.

Solution:

1.	$p \vee q$	(Given)
2.	$q \rightarrow r$	(Given)
3.	$r \rightarrow s$	(Given)
4.1.	$\neg p$	(Assumption)
4.2.	q	(Elim $\vee$: 1, 4.1)
4.3.	r	(Modus Ponens: 4.2, 2)
4.4.	s	(Modus Ponens: 4.3, 3)
4.	$\neg p \rightarrow s$	(Direct proof rule: 4.1-4.4)

8. Formal Proofs: Trickier

(a) Prove that given $p \rightarrow q$, $r \rightarrow p$, $\neg q$, we can conclude $\neg r \vee s$.

Solution:

1.	$p \rightarrow q$	(Given)
2.	$r \rightarrow p$	(Given)
3.	$\neg q$	(Given)
4.	$\neg q \rightarrow \neg p$	(Contrapositive: 1)
5.	$\neg p \rightarrow \neg r$	(Contrapositive: 2)
6.	$\neg p$	(Modus Ponens: 3,4)
7.	$\neg r$	(Modus Ponens: 6,5)
8.	$\neg r \vee s$	(Intro $\vee$: 7)

(b) Prove that for any propositions p, q, we can conclude $p \rightarrow ((q \wedge p) \vee (\neg q \wedge p))$.

Solution:

1.1.	p	(Assumption)
1.2.	$q \vee \neg q$	(Excluded Middle)
1.3.	$(q \vee \neg q) \wedge p$	(Intro $\wedge$: 1.1, 1.2)
1.4.	$(q \wedge p) \vee (\neg q \wedge p)$	(Distributivity: 1.3)
1.	$p \rightarrow ((q \wedge p) \vee (\neg q \wedge p))$	(Direct proof rule: 1.1-1.4)

(c) Prove that for any propositions p, q, r, we can conclude $p \rightarrow (q \rightarrow (r \rightarrow ((p \wedge q) \wedge r)))$.

Solution:

1.1 p (Assumption)

1.2.1 q (Assumption)

1.2.2.1 r (Assumption)

1.2.2.2 $(p \wedge q)$ (Intro $\wedge$: 1.1, 1.2.1)

1.2.2.3 $(p \wedge q) \wedge r$ (Intro $\wedge$: 1.2.2.1, 1.2.2.2)

1.2.2 $r \to ((p \wedge q) \wedge r)$ (Direct proof rule; 1.2.2.1-1.2.2.3)

1.2 $q \to (r \to ((p \wedge q) \wedge r))$ (Direct proof rule; 1.2.1-1.2.2)

1. $p \to (q \to (r \to ((p \wedge q) \wedge r)))$ (Direct proof rule; 1.1-1.2)

9. Challenge: Predicate Translation

Translate "You can fool all of the people some of the time, and you can fool some of the people all of the time, but you can't fool all of the people all of the time" into predicate logic. Then, negate your translation. Then, translate the negation back into English.

Hint: Let the domain of discourse be all people and all times, and let $P(x, y)$ be the statement "You can fool person x at time y". You can get away with not defining any other predicates if you use P.

Solution:

The original statement can thus be translated as $(\forall x \exists y P(x, y)) \wedge (\exists z \forall a P(z, a)) \wedge (\neg \forall b \forall c P(b, c))$.

The negation of this statement, in predicate logic, is $(\exists x \forall y \neg P(x, y)) \vee (\forall z \exists a \neg P(z, a)) \vee (\forall b \forall c P(b, c))$.

The English translation of the negation is "There are some people you can't ever fool, or all people have some time at which you can't fool them, or you can fool everyone at all times".