CSE 390Z: Mathematics for Computation Workshop

Week 2 Workshop Problems Solutions

Conceptual Review

(a) Fill in the following definitions.

Tautology:

Contradiction:

Contingency:

Solution:

Tautology: A proposition that is always true. For example, $p \vee \neg p$.
Contradiction: A proposition that is always false. For example, $p \wedge \neg p$.
Contingency: A proposition that is sometimes true, sometimes false. For example, p.

(b) What is the contrapositive of $p \rightarrow q$? What is the converse of $p \rightarrow q$?

Contrapositive:

Converse:

Solution:

Contrapositive: $\neg q \rightarrow \neg p$. The important thing about the contrapositive is that it's equivalent to the original statement. That is, $p \rightarrow q \equiv \neg q \rightarrow \neg p$.
Converse: $q \rightarrow p$ The important thing about the converse is that it's not necessarily equivalent to the original statement.

(c) What are two different methods to show that two propositions are equivalent?

Solution:

The first method is to write a truth table for each proposition, and check that each row has the same truth value. The second is to use a chain of equivalences that starts at one proposition and ends at the other.

(d) To prove a chain of equivalences, there are many rules we can use (attached to the back of this handout). Fill in some of those rules here.

DeMorgan's Law:

Law of Implication:

Contrapositive:

Solution:

DeMorgan's Laws: $\neg(p \vee q) \equiv \neg p \wedge \neg q$, $\neg(p \wedge q) \equiv \neg p \vee \neg q$
Law of Implication: $p \rightarrow q \equiv \neg p \vee q$
Contrapositive: $p \rightarrow q \equiv \neg q \rightarrow \neg p$

(e) What's the difference between propositional logic, boolean algebra, and circuits?

Solution:

For our purposes, they're all different notation to convey the same underlying meaning.

1. Translation: Running from my problems

Define a set of three atomic propositions, and use them to translate the following sentences.

(i) Whenever it's snowing and it's Friday, I am not going for a run.

(ii) I am going for a run because it is not snowing.

(iii) I am going for a run only if it is not Friday or not snowing.

Solution:

p: I am going for a run
q: It is snowing
r: It is Friday

(i) $(q \wedge r) \to \neg p$

(ii) $\neg q \to p$

(iii) $p \to (\neg r \vee \neg q)$

2. Translation: Age is just a number

Define a set of two atomic propositions, and use them to translate the following sentences.

(i) If Kai is older than thirty, then Kai is older than twenty.

(ii) Kai is older than thirty only if Kai is older than twenty.

(iii) Whenever Kai is older than thirty, Kai is older than twenty.

(iv) Kai being older than twenty is necessary for Kai to be older than thirty.

Solution:

p: Kai is older than thirty
q: Kai is older than twenty

(i) $p \to q$

(ii) $p \to q$

(iii) $p \to q$

(iv) $p \to q$

3. Truth Table

Draw a truth table for $(p \to \neg q) \to (r \oplus q)$

Solution:

p	q	r	$\neg q$	$p \to \neg q$	$r \oplus q$	$(p \to \neg q) \to (r \oplus q)$
T	T	T	F	F	F	T
T	T	F	F	F	T	T
T	F	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	T	F	F	T	T	T
F	F	T	T	T	T	T
F	F	F	T	T	F	F

4. Equivalences: Propositional Logic

Write a chain of logical equivalences to prove the following statements. Note that with propositional logic, you are expected to show all steps, including commutativity and associativity.

(a) $p \to q \equiv \neg(p \wedge \neg q)$

Solution:

These are equivalent. Below is the chain of equivalences.

$$\begin{aligned} p \to q &\equiv \neg p \vee q && \text{Law of Implication} \\ &\equiv \neg p \vee \neg\neg q && \text{Double Negation} \\ &\equiv \neg(p \wedge \neg q) && \text{DeMorgan's Law} \end{aligned}$$

(b) $\neg p \vee ((q \wedge p) \vee (\neg q \wedge p)) \equiv T$

Solution:

$$\begin{aligned} \neg p \vee ((q \wedge p) \vee (\neg q \wedge p)) &\equiv \neg p \vee ((p \wedge q) \vee (\neg q \wedge p)) && \text{Commutativity} \\ &\equiv \neg p \vee ((p \wedge q) \vee (p \wedge \neg q)) && \text{Commutativity} \\ &\equiv \neg p \vee (p \wedge (q \vee \neg q)) && \text{Distributivity} \\ &\equiv \neg p \vee (p \wedge T) && \text{Negation} \\ &\equiv \neg p \vee p && \text{Identity} \\ &\equiv p \vee \neg p && \text{Commutativity} \\ &\equiv T && \text{Negation} \end{aligned}$$

(c) $((p \wedge q) \to r) \equiv (p \to r) \vee (q \to r)$

Solution:

$$\begin{aligned} (p \wedge q) \to r &\equiv \neg(p \wedge q) \vee r && \text{Law of Implication} \\ &\equiv (\neg p \vee \neg q) \vee r && \text{De Morgan's Law} \\ &\equiv (\neg p \vee \neg q) \vee (r \vee r) && \text{Idempotency} \\ &\equiv \neg p \vee (\neg q \vee (r \vee r)) && \text{Associativity} \\ &\equiv \neg p \vee ((\neg q \vee r) \vee r) && \text{Associativity} \\ &\equiv \neg p \vee (r \vee (\neg q \vee r)) && \text{Commutativity} \\ &\equiv \neg p \vee (r \vee (q \to r)) && \text{Law of Implication} \\ &\equiv (\neg p \vee r) \vee (q \to r) && \text{Associativity} \\ &\equiv (p \to r) \vee (q \to r) && \text{Law of Implication} \end{aligned}$$

5. Equivalences: Boolean Algebra

(a) Prove $p' + (p \cdot q) + (q' \cdot p) = 1$ via equivalences.

Solution:

$$
\begin{aligned}
p' + p \cdot q + q' \cdot p &\equiv p' + p \cdot q + p \cdot q' && \text{Commutativity} \\
&\equiv p' + p \cdot (q + q') && \text{Distributivity} \\
&\equiv p' + p \cdot 1 && \text{Complementarity} \\
&\equiv p' + p && \text{Identity} \\
&\equiv p + p' && \text{Commutativity} \\
&\equiv 1 && \text{Complementarity}
\end{aligned}
$$

(b) Prove $(p' + q) \cdot (q + p) = q$ via equivalences.

Solution:

$$
\begin{aligned}
(p' + q) \cdot (q + p) &\equiv (q + p') \cdot (q + p) && \text{Commutativity} \\
&\equiv q + (p' \cdot p) && \text{Distributivity} \\
&\equiv q + (p \cdot p') && \text{Commutativity} \\
&\equiv q + 0 && \text{Complementarity} \\
&\equiv q && \text{Identity}
\end{aligned}
$$

6. Implications and Vacuous Truth

Alice and Bob's teacher says in class "if a number is prime, then the number is odd." Alice and Bob both believe that the teacher is wrong, but for different reasons.

(a) Alice says "9 is odd and not prime, so the implication is false." Is Alice's justification correct? Why or why not?

Solution:

No. She gave an example where the premise (number is prime) is false, and the conclusion is true. This doesn't disprove the claim!

(b) Bob says "2 is prime and not odd, so the implication is false." Is Bob's justification correct? Why or why not?

Solution:

Yes. He gave an example where the premise (number is prime) is true, and the conclusion is false. This does disprove the claim!

(c) Recall that this is the truth table for implications. Which row does Alice's example correspond to? Which row does Bob's example correspond to?

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Solution:

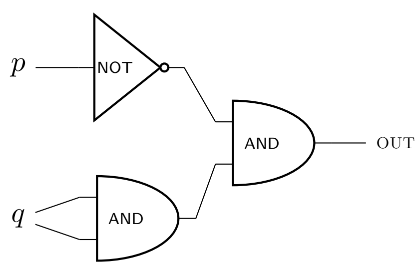
Alice's example corresponds to the row where p is false and q is true. Bob's example corresponds to the row where p is true and q is false.

(d) Observe that in order to show that $p \to q$ is false, you need an example where p is true and q is false. Examples where p is false don't disprove the implication! (Nothing to write for this part).

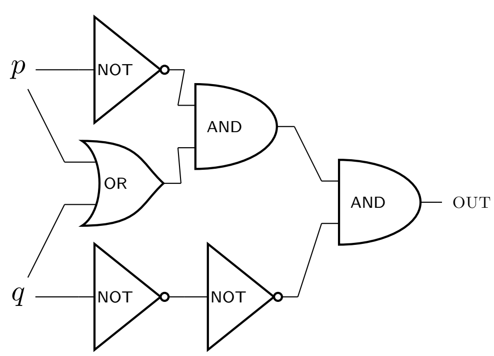
7. Circuits

Convert the following ciruits into logical expressions.

(i)



(ii)



Solution:

(i) $\neg p \wedge (q \wedge q)$

(ii) $((\neg p) \wedge (p \vee q)) \wedge \neg\neg q$

8. Boolean Algebra

Which of the following boolean algebra expressions are equivalent?

(1) $((a' + b') \cdot (a + b))' + (a \cdot b')$

(2) a

(3) b

(4) $a + b'$

(5) $(a' \cdot b)'$

(6) $(a' + b') \cdot a$

(7) $(a \cdot b) + b'$

Solution:

The following are all equivalent, but none of the others are equivalent to any others:

$$(1) \equiv (4) \equiv (5) \equiv (7)$$

This solution first reduces (1) to (7), then (7) to (4). Either (4) or (5) would count as the "simplest" solution, but (7) is not quite simple enough.

$$
\begin{aligned}
((a' + b') \cdot (a + b))' + (a \cdot b') &\equiv (a' + b')' + (a + b)' + (a \cdot b') && \text{DeMorgan's Law} \\
&\equiv (a \cdot b) + (a + b)' + (a \cdot b') && \text{DeMorgan's Law} \\
&\equiv (a \cdot b) + (a' \cdot b') + (a \cdot b') && \text{DeMorgan's Law} \\
&\equiv (a \cdot b) + (a' + a) \cdot b' && \text{Distributivity} \\
&\equiv (a \cdot b) + 1 \cdot b' && \text{Negation} \\
&\equiv (a \cdot b) + b' && \text{Identity} \\
&\equiv (a + b') \cdot (b + b') && \text{Distributivity} \\
&\equiv (a + b') \cdot 1 && \text{Negation} \\
&\equiv a + b' && \text{Identity}
\end{aligned}
$$