0. How Many Ones?

The set $T$ is defined as follows:
- **Base case:** $\epsilon \in T$
- **Recursive Rules:**
  - If $x \in T$, then $11x \in T$
  - If $x \in T$ and $y \in T$, then $x0y \in T$

Given the following recursively defined function
- $\text{numOnes}(\epsilon) = 0$
- $\text{numOnes}(11x) = 2 + \text{numOnes}(x)$
- $\text{numOnes}(x0y) = \text{numOnes}(x) + \text{numOnes}(y)$

Prove that for all strings $n$ in $T$, $\text{numOnes}(n)$ is even.

**Hint:** In structural induction, the structure of your induction mirrors the recursive definition.

**Solution:**
Let $P(n)$ be "$2 \mid \text{numOnes}(n)$". We will show that $P(n)$ is true for all $n \in T$ by structural induction.

**Base Case** ($n = \epsilon$):
$\text{numOnes}(\epsilon) = 0$ definition of numOnes

0 = 2·0 and 2|0 by definition of divides.
Therefore $P(0)$ holds true.

**Induction Hypothesis:** Suppose $P(x)$ and $P(y)$ are true for some arbitrary elements $x, y \in T$.

**Induction Step:**

**Case 1:** $11x$
$\text{numOnes}(11x) = 2 + \text{numOnes}(x)$ by definition of numOnes. By the inductive hypothesis, $2 \mid \text{numOnes}(x)$.

Therefore, by definition of divides $\text{numOnes}(x) = 2z$ for some integer $z$. Thus,

$$\text{numOnes}(11x) = 2 + \text{numOnes}(x) = 2z + 2 = 2(z + 1)$$

Therefore, by definition of divides, $2 \mid \text{numOnes}(11x)$. Therefore, $P(11x)$ holds.

**Case 2:** $x0y$
$\text{numOnes}(x0y) = \text{numOnes}(x) + \text{numOnes}(y)$ by definition of numOnes. By the induction hypothesis, $2 \mid \text{numOnes}(x)$ and $2 \mid \text{numOnes}(y)$.

Therefore, by definition of divides, $\text{numOnes}(x) = 2z$ for some integer $z$ and
numOnes(y) = 2q for some integer q. Thus,

\[ \text{numOnes}(x0y) = \text{numOnes}(x) + \text{numOnes}(y) = 2z + 2q = 2(z + q) \]

Therefore, by definition of divides \(2 \mid \text{numOnes}(x0y)\). Therefore, \(P(x0y)\) holds.

The result follows for all \(n \in T\) by structural induction.

1. **Video Solution**
   Watch this video on the solution after making an initial attempt. Then, answer the following questions.

   (a) What is one thing you took away from the video solution?

   (b) What topic from the quick check or lecture would you most like to review in workshop?