CSE 390Z: Mathematics of Computing Workshop

Week 9 Workshop Solutions

0. Conceptual Review

(a) Regular expression rules:
Basis: ϵ, $\emptyset$, a for $a \in \Sigma$
Recursive: If A, B are regular expressions, $(A \cup B), AB$, and A^* are regular expressions.

(b) Match the relation property to its definition:

Reflexive	$\forall a, b((a, b) \in R \wedge (b, c) \in R) \rightarrow (a, c) \in R)$
Symmetric	$\forall a(a \in A \rightarrow (a, a) \in R)$
Antisymmetric	$\forall a, b((a, b) \in R \rightarrow (b, a) \in R)$
Transitive	$\forall a, b(((a, b) \in R \wedge a \neq b) \rightarrow (b, a) \notin R)$.

Solution:

Reflexive: $\forall a(a \in A \rightarrow (a, a) \in R)$
Symmetric: $\forall a, b((a, b) \in R \rightarrow (b, a) \in R)$
Antisymmetric: $\forall a, b(((a, b) \in R \wedge a \neq b) \rightarrow (b, a) \notin R)$.
Transitive: $\forall a, b((a, b) \in R \wedge (b, c) \in R) \rightarrow (a, c) \in R)$

1. Regular Expressions Warmup

Consider the following Regular Expression (RegEx):

$$1(45 \cup 54)^{\star}1$$

List 5 strings accepted by the RegEx and 5 strings from $T := \{1, 4, 5\}^{\star}$ rejected by the RegEx. Then, summarize this RegEx in your own words.

Solution:

Accepted:

- 1451
- 1541
- 145541
- 1454545451
- 11

Rejected:

- 1
- 1441
- 45
- 14451
- 111

This RegEx accepts exactly those strings that start and end with a 1, and have zero or more pairs of 45 or 54 in the middle.

2. Context Free Grammars Warmup

Consider the following CFG which generates strings from the language V $:= \{0,1,2,3,4\}^*$

$$\mathbf{S} \to 0\mathbf{X}4$$
$$\mathbf{X} \to 1\mathbf{X}3 \mid 2$$

List 5 strings generated by the CFG and 5 strings from V not generated by the CFG. Then, summarize this CFG in your own words.

Solution:

Accepted:

- 024
- 01234
- 0112334
- 011123334
- 01111233334

Rejected:

- ϵ
- 2
- 0244
- 011234
- 10234

This CFG is all strings of the form $0\ 1^m\ 2\ 3^m\ 4$, where $m \geq 0$. That is, it's all strings made of one 0, followed by zero or more 1's, followed by a 2, followed by the same number of 3's as 1's, followed by one 4.

3. Simplify the RegEx

Consider the following Regular Expression (RegEx):

$$0^\star(0 \cup 1)^\star((01) \cup (11) \cup (10) \cup (00))1^\star(0 \cup 1)^\star$$

List 3 strings accepted by the RegEx and 3 strings from $S := \{0,1\}^\star$ rejected by the RegEx. Then, summarize this RegEx in your own words and write a simpler RegEx that accepts exactly the same set of strings.

Solution:

Accepted:

- 01
- 10
- 10100100101

Rejected:

- ϵ
- 0
- 1

This RegEx accepts all binary strings that are 2 or more characters long. A simpler RegEx for this is $(0\cup1)(0\cup 1)(0\cup1)^\star$.

4. Constructing RegExs and CFGs

For each of the following, construct a regular expression and CFG for the specified language.

(a) Strings from the language $S := \{a\}^*$ with an even number of a's.

Solution:

$$(aa)^*$$

$$\mathbf{S} \rightarrow aa\mathbf{S}|\varepsilon$$

(b) Strings from the language $S := \{a, b\}^*$ with an even number of a's.

Solution:

$$b^*(b^*ab^*ab^*)^*$$

$$\mathbf{S} \rightarrow bS|aSaS|\epsilon$$

(c) Strings from the language $S := \{a, b\}^*$ with odd length.

Solution:

$$(aa \cup ab \cup ba \cup bb)^*(a \cup b)$$

$$\mathbf{S} \rightarrow \mathbf{CS}|a|b$$
$$\mathbf{C} \rightarrow aa\mathbf{C}|ab\mathbf{C}|ba\mathbf{C}|bb\mathbf{C}|\varepsilon$$

(d) (Challenge) Strings from the language $S := \{a, b\}^*$ with an even number of a's or an odd number of b's.

Solution:

$$b^*(b^*ab^*ab^*)^* \cup (a^* \cup a^*ba^*ba^*)^*b(a^* \cup a^*ba^*ba^*)^*$$

$$\mathbf{S} \rightarrow \mathbf{E}|\mathbf{O}b\mathbf{O}$$
$$\mathbf{E} \rightarrow \mathbf{EE}|a\mathbf{E}a|b|\varepsilon$$
$$\mathbf{O} \rightarrow \mathbf{OO}|b\mathbf{O}b|a|\varepsilon$$

5. Relations Examples

(a) Consider the relation $R \subseteq \mathbb{Z} \times \mathbb{Z}$ defined by $(a, b) \in R$ iff $a \leq b + 1$. List 3 pairs of integers that are in R, and 3 pairs of integers that are not.

Solution:

In R: $(0, 0), (1, 0), (-1, 0)$
Not in R: $(2, 0), (3, 0), (17, 5)$

(b) Consider the relation $R \subseteq \mathbb{Z} \times \mathbb{Z}$ defined by $(a, b) \in R$ iff $a \leq b+1$. Determine if R is reflexive, symmetric, antisymmetric, and/or transitive. If a relation has a property, explain why. If not, state a counterexample.

Solution:

- Reflexive: Yes. For any integer a, it is true that $a \leq a + 1$. So $(a, a) \in R$.
- Symmetric: No. For example, $(0, 20) \in R$ but $(20, 0) \notin R$.
- Antisymmetric: No. For example $(0, 1) \in R$ and $(1, 0) \in R$.
- Transitive: No. For example $(2, 1) \in R$ and $(1, 0) \in R$, but $(2, 0) \notin R$.

6. Relations Proofs

Suppose that $R, S \subseteq \mathbb{Z} \times \mathbb{Z}$ are relations.

(a) Prove or disprove: If R and S are transitive, $R \cup S$ is transitive.

Solution:

False. Let $R = \{(1, 2)\}$, $S = \{(2, 1)\}$. By definition, R and S are transitive. By definition of union, $R \cup S = \{(1, 2), (2, 1)\}$. However, if $R \cup S$ was transitive, we would require $(1, 1)$ to be in $R \cup S$, because (1,2) and (2,1) is in $R \cup S$. However, this is not the case. Therefore the claim is false.

(b) Prove or disprove: If R is symmetric, $\overline{R}$ (the complement of R) is symmetric.

Solution:

True. Since R is symmetric, we know the following.

$$\forall a \forall b \ [(a, b) \in R \rightarrow (b, a) \in R]$$

Taking the contrapositive, this is equivalent to:

$$\forall a \forall b \ [(b, a) \notin R \rightarrow (a, b) \notin R]$$

By the definition of complement, this is equivalent to:

$$\forall a \forall b \ [(b, a) \in \overline{R} \rightarrow (a, b) \in \overline{R}]$$

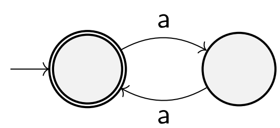
This is the definition of $\overline{R}$ being symmetric.

7. Constructing DFAs

For each of the following, construct a DFA for the specified language.

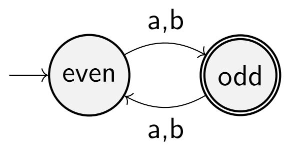
(a) Strings from the language $\Sigma := \{a\}^*$ with an even number of a's.

Solution:



(b) Strings from the language $\Sigma = \{a, b\}$ with odd length.

Solution:



(c) Strings from the language $\Sigma = \{a, b\}$ with an even number of a's or an odd number of b's.

Solution: