0. Complete the Induction Proof

Consider the function \( f(n) \) defined for integers \( n \geq 1 \) as follows:

\[
\begin{align*}
  f(1) &= 1 \\
  f(2) &= 4 \\
  f(3) &= 9 \\
  f(n) &= f(n-1) - f(n-2) + f(n-3) + 2(2n-3) & \text{for } n \geq 4
\end{align*}
\]

Prove by strong induction that for all \( n \geq 1 \), \( f(n) = n^2 \).

Complete the induction proof below:

Solution:

Let \( P(n) \) be defined as " \( f(n) = n^2 \)". We will prove \( P(n) \) is true for all integers \( n \geq 1 \) by strong induction. 

\( n = 1, 2, 3 \):

- \( n = 1 \): \( f(1) = 1 = 1^2 \).
- \( n = 2 \): \( f(2) = 4 = 2^2 \).
- \( n = 3 \): \( f(3) = 9 = 3^2 \)

So the base cases hold.

Suppose for some arbitrary integer \( k \geq 3 \), \( P(1) \land \ldots \land P(k) \) hold.

\[ \text{Goal: Show } P(k+1), \text{ i.e. show that } f(k+1) = (k+1)^2. \]

\[
\begin{align*}
  f(k+1) &= f(k+1-1) - f(k+1-2) + f(k+1-3) + 2(2(k+1)-3) & \text{Definition of } f \\
        &= f(k) - f(k-1) + f(k-2) + 2(2k-1) \\
        &= k^2 - (k-1)^2 + (k-2)^2 + 2(2k-1) & \text{By IH} \\
        &= k^2 - k^2 + k^2 + (k^2 - 4k + 4) + 4k - 2 \\
        &= (k^2 - k^2 + k^2) + (2k - 4k + 4k) + (-1 + 4 - 2) \\
        &= k^2 + 2k + 1 \\
        &= (k+1)^2 
\end{align*}
\]

So \( P(k+1) \) holds.

So by strong induction, \( P(n) \) is true for all integers \( n \geq 1 \).
1. Induction: Another Inequality

Prove by induction on \( n \) that for all integers \( n \geq 4 \) the inequality \( n! > 2^n \) is true.

**Solution:**

Let \( P(n) \) be "\( n! > 2^n \)." We will prove \( P(n) \) is true for all \( n \in \mathbb{N}, n \geq 4 \), by induction.

**Base Case:** \( n = 4 \): \( 4! = 24 \) and \( 2^4 = 16 \), since \( 24 > 16 \), \( P(4) \) is true.

**Inductive Hypothesis:** Suppose that \( P(k) \) is true for some arbitrary integer \( k \in \mathbb{N}, k \geq 4 \).

**Inductive Step:**

<table>
<thead>
<tr>
<th>Goal: Show ( P(k + 1) ), i.e. show ( (k + 1)! &gt; 2^{k+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (k + 1)! = k! \cdot (k + 1) )</td>
</tr>
<tr>
<td>( &gt; 2^k \cdot (k + 1) ) (By I.H., ( k! &gt; 2^k ))</td>
</tr>
<tr>
<td>( &gt; 2^k \cdot 2 ) (Since ( k \geq 4 ), so ( k + 1 \geq 5 &gt; 2 ))</td>
</tr>
<tr>
<td>( = 2^{k+1} )</td>
</tr>
</tbody>
</table>

**Conclusion:** So by induction, \( P(n) \) is true for all \( n \in \mathbb{N}, n \geq 4 \).
2. Induction: Divides

Prove that \(9 \mid (n^3 + (n + 1)^3 + (n + 2)^3)\) for all \(n > 1\) by induction.

Solution:

Let \(P(n)\) be “\(9 \mid n^3 + (n + 1)^3 + (n + 2)^3\)”. We will prove \(P(n)\) for all integers \(n > 1\) by induction.

**Base Case** \((n = 2)\): \(2^3 + (2 + 1)^3 + (2 + 2)^3 = 8 + 27 + 64 = 99 = 9 \cdot 11\), so \(9 \mid 2^3 + (2 + 1)^3 + (2 + 2)^3\), so \(P(2)\) holds.

**Inductive Hypothesis**: Assume that \(9 \mid k^3 + (k + 1)^3 + (k + 2)^3\) for an arbitrary integer \(k > 1\). Note that this is equivalent to assuming that \(k^3 + (k + 1)^3 + (k + 2)^3 = 9j\) for some integer \(j\) by the definition of divides.

**Inductive Step**: \(\boxed{\text{Goal: Show } 9 \mid (k + 1)^3 + (k + 2)^3 + (k + 3)^3}\)

\[
(k + 1)^3 + (k + 2)^3 + (k + 3)^3 = (k^2 + 6k + 9)(k + 3) + (k + 1)^3 + (k + 2)^3 \\
= (k^3 + 6k^2 + 9k + 3k^2 + 18k + 27) + (k + 1)^3 + (k + 2)^3 \\
= 9k^2 + 27k + 27 + k^3 + (k + 1)^3 + (k + 2)^3 \\
= 9k^2 + 27k + 27 + 9j \\
= 9(k^2 + 3k + 3 + j) \\
= 9(k^2 + 3k + 3 + j) \\
\]

Since \(k\) and \(j\) are integers, \(k^2 + 3k + 3 + j\) is also an integer. Therefore, by the definition of divides, \(9 \mid (k + 1)^3 + (k + 2)^3 + (k + 3)^3\), so \(P(k) \rightarrow P(k + 1)\) for an arbitrary integer \(k > 1\).

**Conclusion**: \(P(n)\) holds for all integers \(n > 1\) by induction.
3. **Strong Induction: Stamp Collection**

A store sells 3 cent and 5 cent stamps. Use strong induction to prove that you can make exactly \( n \) cents worth of stamps for all \( n \geq 10 \).

**Hint:** you’ll need multiple base cases for this - think about how many steps back you need to go for your inductive step.

**Solution:**

Let \( P(n) \) be defined as "You can buy exactly \( n \) cents of stamps". We will prove \( P(n) \) is true for all integers \( n \geq 10 \) by strong induction.

**Base Cases:** \( (n = 10, 11, 12) \):

- \( n = 10 \): 10 cents of stamps can be made from two 5 cent stamps.
- \( n = 11 \): 11 cents of stamps can be made from one 5 cent and two 3 cent stamps.
- \( n = 12 \): 12 cents of stamps can be made from four 3 cent stamps.

**Inductive Hypothesis:** Suppose for some arbitrary integer \( k \geq 12 \), \( P(10) \land P(11) \land ... \land P(k) \) holds.

**Inductive Step:**

**Goal:** Show \( P(k+1) \), i.e. show that we can make \( k+1 \) cents in stamps.

We want to buy \( k + 1 \) cents in stamps. By the I.H., we can buy exactly \( (k + 1) - 3 = k - 2 \) cents in stamps. Then, we can add another 3 cent stamp in order to buy \( k + 1 \) cents in stamps, so \( P(k+1) \) is true.

**Note:** How did we decide how many base cases to have? Well, we wanted to be able to assume \( P(k-2) \), and add 3 to achieve \( P(k+1) \). Therefore we needed to be able to assume that \( k - 2 \geq 10 \). Adding 2 to both sides, we needed to be able to assume that \( k \geq 12 \). So, we have to prove the base cases up to 12, that is: 10, 11, 12.

Another way to think about this is that we had to use a fact from 3 steps back from \( k + 1 \) to \( k - 2 \) in the IS, so we needed 3 base cases.

**Conclusion:** So by strong induction, \( P(n) \) is true for all integers \( n \geq 10 \).
4. Strong Induction: Functions
Let a function $f$ be defined by:

- $f(1) = 0$
- $f(2) = 12$
- $f(n) = 4 \cdot f(n - 1) - 3 \cdot f(n - 2)$ for $n \geq 3$

Prove that $f(n) = 2 \cdot 3^n - 6$ for any positive integer $n$.

**Solution:**

Let $P(n)$ be the claim that $f(n) = 2 \cdot 3^n - 6$. We will prove $P(n)$ true for all integers $n \geq 1$ using strong induction.

**Base Case:**

- For $n = 1$, $2 \cdot 3^1 - 6 = 2 \cdot 3 - 6 = 6 - 6 = 0 = f(1)$, so $P(1)$ holds.
- For $n = 2$, $2 \cdot 3^2 - 6 = 2 \cdot 9 - 6 = 18 - 6 = 12 = f(2)$, so $P(2)$ holds.

**Inductive Hypothesis:** Suppose that $P(j)$ holds all $1 \leq j \leq k$ for some arbitrary positive integer $k \geq 2$.

**Inductive Step:**

**Goal:** Show $P(k+1)$, i.e. $f(k+1) = 2 \cdot 3^{k+1} - 6$.

\[
\begin{align*}
 f(k + 1) & = 4 \cdot f((k + 1) - 1) - 3 \cdot f((k + 1) - 2) & \text{Definition of } f \\
 & = 4 \cdot f(k) - 3 \cdot f(k - 1) \\
 & = 4 \cdot (2 \cdot 3^k - 6) - 3 \cdot (2 \cdot 3^{k-1} - 6) & \text{I.H.} \\
 & = 8 \cdot 3^k - 24 - 6 \cdot 3^{k-1} + 18 \\
 & = 8 \cdot 3^k - 6 \cdot 3^{k-1} - 6 \\
 & = 8 \cdot 3^k - 2 \cdot 3^k - 6 \\
 & = 6 \cdot 3^k - 6 \\
 & = 2 \cdot 3^{k+1} - 6 \\
\end{align*}
\]

Thus, $P(k+1)$ holds.

**Conclusion:** Therefore, by the principles of strong induction, $P(n)$ holds for all positive integers $n$. 


5. Strong Induction: Collecting Candy

A store sells candy in packs of 4 and packs of 7. Let \( P(n) \) be defined as "You are able to buy \( n \) packs of candy". For example, \( P(3) \) is not true, because you cannot buy exactly 3 packs of candy from the store. However, it turns out that \( P(n) \) is true for any \( n \geq 18 \). Use strong induction on \( n \) to prove this.

Hint: you'll need multiple base cases for this - think about how many steps back you need to go for your inductive step.

Solution:

Let \( P(n) \) be defined as "You are able to buy \( n \) packs of candy". We will prove \( P(n) \) is true for all integers \( n \geq 18 \) by strong induction.

**Base Cases:** \( n = 18, 19, 20, 21 \):

- \( n = 18 \): 18 packs of candy can be made up of 2 packs of 7 and 1 pack of 4 (\( 18 = 2 \times 7 + 1 \times 4 \)).
- \( n = 19 \): 19 packs of candy can be made up of 1 pack of 7 and 3 packs of 4 (\( 19 = 1 \times 7 + 3 \times 4 \)).
- \( n = 20 \): 20 packs of candy can be made up of 5 packs of 4 (\( 20 = 5 \times 4 \)).
- \( n = 21 \): 21 packs of candy can be made up of 3 packs of 7 (\( 21 = 3 \times 7 \)).

**Inductive Hypothesis:** Suppose for some arbitrary integer \( k \geq 21 \), \( P(18) \land \ldots \land P(k) \) hold.

**Inductive Step:**

**Goal:** Show \( P(k + 1) \), i.e. show that we can buy \( k + 1 \) packs of candy.

We want to buy \( k + 1 \) packs of candy. By the I.H., we can buy exactly \( k - 3 \) packs, so we can add another pack of 4 packs in order to buy \( k + 1 \) packs of candy, so \( P(k + 1) \) is true.

**Note:** How did we decide how many base cases to have? Well, we wanted to be able to assume \( P(k - 3) \), and add 4 to achieve \( P(k + 1) \). Therefore we needed to be able to assume that \( k - 3 \geq 18 \). Adding 3 to both sides, we needed to be able to assume that \( k \geq 21 \). So, we have to prove the base cases up to 21, that is: 18, 19, 20, 21.

Another way to think about this is that we had to use a fact from 4 steps back from \( k + 1 \) to \( k - 3 \) in the IS, so we needed 4 base cases.

**Conclusion:** So by strong induction, \( P(n) \) is true for all integers \( n \geq 18 \).
6. Structural Induction: a’s and b’s

Define a set $S$ of character strings over the alphabet $\{a, b\}$ by:

- $a$ and $ab$ are in $S$
- If $x \in S$ and $y \in S$, then $axb \in S$ and $xy \in S$

Prove by induction that every string in $S$ has at least as many $a$’s as it does $b$’s.

**Solution:**

Let $P(s)$ be the claim that a string has at least many $a$’s as it does $b$’s. We will prove $P(s)$ true for all strings $s \in S$ using structural induction.

**Base Case:**

- Consider $s = a$: there is one $a$ and zero $b$’s, so $P(a)$ holds.
- Consider $s = ab$: there is one $a$ and one $b$, so $P(ab)$ holds.

Let $t$ be an arbitrary string in $S$ that is not one of the base cases. Then, by the exclusion rule, it must be that $t = axb$ or $t = xy$ for some $x, y \in S$.

**Inductive Hypothesis:** Suppose $P(x)$ and $P(y)$ hold.

**Inductive Step:**

**Goal:** Prove $P(axb)$ and $P(xy)$

First, we consider $axb$. We are adding one $a$ and one $b$ to $x$. Per the IH, $x$ must have at least as many $a$’s as it does $b$’s. Therefore, since adding one $a$ and one $b$ does not change the difference in the number of $a$’s and $b$’s, $axb$ must have at least many $a$’s as it does $b$’s. Thus, $P(axb)$ holds.

Second, we consider $xy$. Let $m, n$ represent the number of $a$’s in $x$ and $y$ respectively. Similarly, let $i, j$ represent the number of $b$’s in $x$ and $y$. Per the IH, we know that $m \geq i$ and $n \geq j$. Adding these together, we see $m + n \geq i + j$. Therefore, $xy$ must have at least as many $a$’s (i.e., $m + n$ a’s) as it does $b$’s (i.e., $i + j$ b’s). Thus, $P(xy)$ holds.

So, $P(t)$ holds in both cases. **Conclusion:** Therefore, per the principles of structural induction, $P(s)$ holds for all strings in $S$.  

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7. Structural Induction: Divisible by 4

Define a set $\mathcal{B}$ of numbers by:

- $4$ and $12$ are in $\mathcal{B}$
- If $x \in \mathcal{B}$ and $y \in \mathcal{B}$, then $x + y \in \mathcal{B}$ and $x - y \in \mathcal{B}$

Prove by induction that every number in $\mathcal{B}$ is divisible by $4$.

**Solution:**

Let $P(b)$ be the claim that $4 \mid b$. We will prove $P$ true for all numbers $b \in \mathcal{B}$ by structural induction.

**Base Case:**

- $4 \mid 4$ is trivially true, so $P(4)$ holds.
- $12 = 3 \cdot 4$, so $4 \mid 12$ and $P(12)$ holds.

Let $t$ be an arbitrary element from $\mathcal{B}$ that is not from the base cases. Then, by the exclusion rule, it must be that $t = x + y$ or $t = x - y$ for some $x, y \in \mathcal{B}$.

**Inductive Hypothesis:** Suppose $P(x)$ and $P(y)$.

**Inductive Step:**

**Goal:** Prove $P(x+y)$ and $P(x-y)$

Per the IH, $4 \mid x$ and $4 \mid y$. By the definition of divides, $x = 4k$ and $y = 4j$ for some integers $k, j$. Then, $x + y = 4k + 4j = 4(k + j)$. Since integers are closed under addition, $k + j$ is an integer, so $4 \mid x + y$ and $P(x+y)$ holds.

Similarly, $x - y = 4k - 4j = 4(k - j) = 4(k + (-1 \cdot j))$. Since integers are closed under addition and multiplication, and $-1$ is an integer, we see that $k - j$ must be an integer. Therefore, by the definition of divides, $4 \mid x - y$ and $P(x - y)$ holds.

So, $P(t)$ holds in both cases.

**Conclusion:** Therefore, $P(b)$ holds for all numbers $b \in \mathcal{B}$.