CSE 390Z: Mathematics for Computation Workshop

Week 7 Workshop Solutions

0. Complete the Induction Proof

Consider the function $f(n)$ defined for integers $n \geq 1$ as follows:
$f(1) = 1$ for $n = 1$
$f(2) = 4$ for $n = 2$
$f(3) = 9$ for $n = 3$
$f(n) = f(n-1) - f(n-2) + f(n-3) + 2(2n-3)$ for $n \geq 4$

Prove by strong induction that for all $n \geq 1$, $f(n) = n^2$.

Complete the induction proof below:

Solution:

Let P(n) be defined as " $f(n) = n^2$". We will prove $P(n)$ is true for all integers $n \geq 1$ by strong induction.

$(n = 1, 2, 3)$**:**

- $n = 1$: $f(1) = 1 = 1^2$.
- $n = 2$: $f(2) = 4 = 2^2$.
- $n = 3$: $f(3) = 9 = 3^2$

So the base cases hold.

Suppose for some arbitrary integer $k \geq 3$, P$(1) \wedge \ldots \wedge$P(k) hold.

Goal: Show $P(k+1)$, i.e. show that $f(k+1) = (k+1)^2$.

$$
\begin{aligned}
f(k+1) &= f(k+1-1) - f(k+1-2) + f(k+1-3) + 2(2(k+1)-3) && \text{Definition of f} \\
&= f(k) - f(k-1) + f(k-2) + 2(2k-1) \\
&= k^2 - (k-1)^2 + (k-2)^2 + 2(2k-1) && \text{By IH} \\
&= k^2 - (k^2 - 2k + 1) + (k^2 - 4k + 4) + 4k - 2 \\
&= (k^2 - k^2 + k^2) + (2k - 4k + 4k) + (-1 + 4 - 2) \\
&= k^2 + 2k + 1 \\
&= (k+1)^2
\end{aligned}
$$

So P$(k+1)$ holds.

So by strong induction, P(n) is true for all integers $n \geq 1$.

1. Induction: Another Inequality

Prove by induction on n that for all integers $n \geq 4$ the inequality $n! > 2^n$ is true.

Solution:

Let $P(n)$ be "$n! > 2^n$". We will prove $P(n)$ is true for all $n \in \mathbb{N}$, $n \geq 4$, by induction.

Base Case: (n = 4)**:** $4! = 24$ and $2^4 = 16$, since $24 > 16$, $P(4)$ is true.

Inductive Hypothesis: Suppose that $P(k)$ is true for some arbitrary integer $k \in \mathbb{N}$, $k \geq 4$.

Inductive Step:

Goal: Show $P(k+1)$, i.e. show $(k+1)! > 2^{k+1}$

$$
\begin{aligned}
(k+1)! &= k! \cdot (k+1) \\
&> 2^k \cdot (k+1) && \text{(By I.H., } k! > 2^k\text{)} \\
&> 2^k \cdot 2 && \text{(Since } k \geq 4 \text{, so } k+1 \geq 5 > 2\text{)} \\
&= 2^{k+1}
\end{aligned}
$$

Conclusion: So by induction, $P(n)$ is true for all $n \in \mathbb{N}$, $n \geq 4$.

2. Induction: Divides

Prove that $9 \mid (n^3 + (n+1)^3 + (n+2)^3)$ for all $n > 1$ by induction.

Solution:

Let $P(n)$ be "$9 \mid n^3 + (n+1)^3 + (n+2)^3$". We will prove $P(n)$ for all integers $n > 1$ by induction.

Base Case $(n = 2)$**:** $2^3 + (2+1)^3 + (2+2)^3 = 8 + 27 + 64 = 99 = 9 \cdot 11$, so $9 \mid 2^3 + (2+1)^3 + (2+2)^3$, so $P(2)$ holds.

Inductive Hypothesis: Assume that $9 \mid k^3 + (k+1)^3 + (k+2)^3$ for an arbitrary integer $k > 1$. Note that this is equivalent to assuming that $k^3 + (k+1)^3 + (k+2)^3 = 9j$ for some integer j by the definition of divides.

Inductive Step: Goal: Show $9 \mid (k+1)^3 + (k+2)^3 + (k+3)^3$

$$
\begin{aligned}
(k+1)^3 + (k+2)^3 + (k+3)^3 &= (k^2 + 6k + 9)(k+3) + (k+1)^3 + (k+2)^3 && \text{[expanding trinomial]} \\
&= (k^3 + 6k^2 + 9k + 3k^2 + 18k + 27) + (k+1)^3 + (k+2)^3 && \text{[expanding binomial]} \\
&= 9k^2 + 27k + 27 + k^3 + (k+1)^3 + (k+2)^3 && \text{[adding like terms]} \\
&= 9k^2 + 27k + 27 + 9j && \text{[by I.H.]} \\
&= 9(k^2 + 3k + 3 + j) && \text{[factoring out 9]}
\end{aligned}
$$

Since k and j are integers, $k^2 + 3k + 3 + j$ is also an integer. Therefore, by the definition of divides, $9 \mid (k+1)^3 + (k+2)^3 + (k+3)^3$, so $P(k) \to P(k+1)$ for an arbitrary integer $k > 1$.

Conclusion: $P(n)$ holds for all integers $n > 1$ by induction.

3. Strong Induction: Stamp Collection

A store sells 3 cent and 5 cent stamps. Use strong induction to prove that you can make exactly n cents worth of stamps for all $n \geq 10$.

Hint: you'll need multiple base cases for this - think about how many steps back you need to go for your inductive step.

Solution:

Let P(n) be defined as "You can buy exactly n cents of stamps". We will prove $P(n)$ is true for all integers $n \geq 10$ by strong induction.

Base Cases: $(n = 10, 11, 12)$**:**

- $n = 10$: 10 cents of stamps can be made from two 5 cent stamps.
- $n = 11$: 11 cents of stamps can be made from one 5 cent and two 3 cent stamps.
- $n = 12$: 12 cents of stamps can be made from four 3 cent stamps.

Inductive Hypothesis: Suppose for some arbitrary integer $k \geq 12$, P(10) $\wedge$P(11) $\wedge$... $\wedge$P(k) holds.

Inductive Step:

Goal: Show $P(k+1)$, i.e. show that we can make $k+1$ cents in stamps.

We want to buy $k + 1$ cents in stamps. By the I.H., we can buy exactly $(k + 1) - 3 = k - 2$ cents in stamps. Then, we can add another 3 cent stamp in order to buy $k + 1$ cents in stamps, so P$(k + 1)$ is true.

Note: How did we decide how many base cases to have? Well, we wanted to be able to assume P$(k-2)$, and add 3 to achieve P$(k + 1)$. Therefore we needed to be able to assume that $k - 2 \geq 10$. Adding 2 to both sides, we needed to be able to assume that $k \geq 12$. So, we have to prove the base cases up to 12, that is: $10, 11, 12$.

Another way to think about this is that we had to use a fact from 3 steps back from $k + 1$ to $k - 2$ in the IS, so we needed 3 base cases.

Conclusion: So by strong induction, P(n) is true for all integers $n \geq 10$.

4. Strong Induction: Functions

Let a function f be defined by:

- $f(1) = 0$
- $f(2) = 12$
- $f(n) = 4 \cdot f(n-1) - 3 \cdot f(n-2)$ for $n \geq 3$

Prove that $f(n) = 2 \cdot 3^n - 6$ for any positive integer n.

Solution:

Let $P(n)$ be the claim that $f(n) = 2 \cdot 3^n - 6$. We will prove $P(n)$ true for all integers $n \geq 1$ using strong induction.

Base Case:

- For $n = 1$, $2 \cdot 3^1 - 6 = 2 \cdot 3 - 6 = 6 - 6 = 0 = f(1)$, so $P(1)$ holds.
- For $n = 2$, $2 \cdot 3^2 - 6 = 2 \cdot 9 - 6 = 18 - 6 = 12 = f(2)$, so $P(2)$ holds.

Inductive Hypothesis: Suppose that $P(j)$ holds all $1 \leq j \leq k$ for some arbitrary positive integer $k \geq 2$.

Inductive Step:

Goal: Show $P(k+1)$, i.e. $f(k+1) = 2 \cdot 3^{k+1} - 6$.

$$
\begin{aligned}
f(k+1) &= 4 \cdot f((k+1)-1) - 3 \cdot f((k+1)-2) && \text{Definition of } f \\
&= 4 \cdot f(k) - 3 \cdot f(k-1) \\
&= 4 \cdot (2 \cdot 3^k - 6) - 3 \cdot (2 \cdot 3^{k-1} - 6) && \text{I.H.} \\
&= 8 \cdot 3^k - 24 - 6 \cdot 3^{k-1} + 18 \\
&= 8 \cdot 3^k - 6 \cdot 3^{k-1} - 6 \\
&= 8 \cdot 3^k - 2 \cdot 3^k - 6 \\
&= 6 \cdot 3^k - 6 \\
&= 2 \cdot 3^{k+1} - 6
\end{aligned}
$$

Thus, $P(k+1)$ holds.

Conclusion: Therefore, by the principles of strong induction, $P(n)$ holds for all positive integers n.

5. Strong Induction: Collecting Candy

A store sells candy in packs of 4 and packs of 7. Let P(n) be defined as "You are able to buy n packs of candy". For example, $P(3)$ is not true, because you cannot buy exactly 3 packs of candy from the store. However, it turns out that P(n) is true for any $n \geq 18$. Use strong induction on n to prove this.

Hint: you'll need multiple base cases for this - think about how many steps back you need to go for your inductive step.

Solution:

Let P(n) be defined as "You are able to buy n packs of candy". We will prove $P(n)$ is true for all integers $n \geq 18$ by strong induction.

Base Cases: ($n = 18, 19, 20, 21$)**:**

- $n = 18$: 18 packs of candy can be made up of 2 packs of 7 and 1 pack of 4 ($18 = 2 * 7 + 1 * 4$).
- $n = 19$: 19 packs of candy can be made up of 1 pack of 7 and 3 packs of 4 ($19 = 1 * 7 + 3 * 4$).
- $n = 20$: 20 packs of candy can be made up of 5 packs of 4 ($20 = 5 * 4$).
- $n = 21$: 21 packs of candy can be made up of 3 packs of 7 ($21 = 3 * 7$).

Inductive Hypothesis: Suppose for some arbitrary integer $k \geq 21$, P(18) $\wedge$... $\wedge$P(k) hold.

Inductive Step:

Goal: Show $P(k+1)$, i.e. show that we can buy $k+1$ packs of candy.

We want to buy $k+1$ packs of candy. By the I.H., we can buy exactly $k-3$ packs, so we can add another pack of 4 packs in order to buy $k+1$ packs of candy, so P($k+1$) is true.

Note: How did we decide how many base cases to have? Well, we wanted to be able to assume P($k-3$), and add 4 to achieve P($k+1$). Therefore we needed to be able to assume that $k-3 \geq 18$. Adding 3 to both sides, we needed to be able to assume that $k \geq 21$. So, we have to prove the base cases up to 21, that is: $18, 19, 20, 21$.

Another way to think about this is that we had to use a fact from 4 steps back from $k+1$ to $k-3$ in the IS, so we needed 4 base cases.

Conclusion: So by strong induction, P(n) is true for all integers $n \geq 18$.

6. Structural Induction: a's and b's

Define a set $\mathcal{S}$ of character strings over the alphabet $\{a, b\}$ by:

- a and ab are in $\mathcal{S}$
- If $x \in \mathcal{S}$ and $y \in \mathcal{S}$, then $axb \in \mathcal{S}$ and $xy \in \mathcal{S}$

Prove by induction that every string in $\mathcal{S}$ has at least as many a's as it does b's.

Solution:

Let $P(s)$ be the claim that a string has at least many a's as it does b's. We will prove $P(s)$ true for all strings $s \in \mathcal{S}$ using structural induction.

Base Case:

- Consider $s = a$: there is one a and zero b's, so $P(a)$ holds.
- Consider $s = ab$: there is one a and one b, so $P(ab)$ holds.

Let t be an arbitrary string in $\mathcal{S}$ that is not one of the base cases. Then, by the exclusion rule, it must be that $t = axb$ or $t = xy$ for some $x, y \in \mathcal{S}$.

Inductive Hypothesis: Suppose $P(x)$ and $P(y)$ hold.

Inductive Step:

Goal: Prove $P(axb)$ and $P(xy)$

First, we consider axb. We are adding one a and one b to x. Per the IH, x must have at least as many a's as it does b's. Therefore, since adding one a and one b does not change the *difference* in the number of a's and b's, axb must have at least many a's as it does b's. Thus, $P(axb)$ holds.

Second, we consider xy. Let m, n represent the number of a's in x and y respectively. Similarly, let i, j represent the number of b's in x and y. Per the IH, we know that $m \geq i$ and $n \geq j$. Adding these together, we see $m + n \geq i + j$. Therefore, xy must have at least as many a's (i.e., $m + n$ a's) as it does b's (i.e., $i + j$ b's). Thus, $P(xy)$ holds.

So, $P(t)$ holds in both cases. **Conclusion:** Therefore, per the principles of structural induction, $P(s)$ holds for all strings in $\mathcal{S}$.

7. Structural Induction: Divisible by 4

Define a set $\mathfrak{B}$ of numbers by:

- 4 and 12 are in $\mathfrak{B}$
- If $x \in \mathfrak{B}$ and $y \in \mathfrak{B}$, then $x + y \in \mathfrak{B}$ and $x - y \in \mathfrak{B}$

Prove by induction that every number in $\mathfrak{B}$ is divisible by 4.

Solution:

Let $P(b)$ be the claim that $4 \mid b$. We will prove P true for all numbers $b \in \mathfrak{B}$ by structural induction.
Base Case:

- $4 \mid 4$ is trivially true, so $P(4)$ holds.
- $12 = 3 \cdot 4$, so $4 \mid 12$ and $P(12)$ holds.

Let t be an arbitrary element from $\mathfrak{B}$ that is not from the base cases. Then, by the exclusion rule, it must be that $t = x + y$ or $t = x - y$ for some $x, y \in \mathfrak{B}$.
Inductive Hypothesis: Suppose $P(x)$ and $P(y)$.
Inductive Step:

Goal: Prove $P(x+y)$ and $P(x-y)$

Per the IH, $4 \mid x$ and $4 \mid y$. By the definition of divides, $x = 4k$ and $y = 4j$ for some integers k, j. Then, $x + y = 4k + 4j = 4(k + j)$. Since integers are closed under addition, $k + j$ is an integer, so $4 \mid x + y$ and $P(x + y)$ holds.
Similarly, $x - y = 4k - 4j = 4(k - j) = 4(k + (-1 \cdot j))$. Since integers are closed under addition and multiplication, and -1 is an integer, we see that $k - j$ must be an integer. Therefore, by the definition of divides, $4 \mid x - y$ and $P(x - y)$ holds.
So, $P(t)$ holds in both cases.
Conclusion: Therefore, $P(b)$ holds for all numbers $b \in \mathfrak{B}$.