CSE 390Z: Mathematics for Computation Workshop

Week 2 Workshop Problems Solutions

Conceptual Review

(a) What is the contrapositive of $p \to q$? What is the converse of $p \to q$?

Contrapositive:

Converse:

Solution:

Contrapositive: $\neg q \to \neg p$. The important thing about the contrapositive is that it's equivalent to the original statement. That is, $p \to q \equiv \neg q \to \neg p$.
Converse: $q \to p$ The important thing about the converse is that it's not necessarily equivalent to the original statement.

(b) What are two different methods to show that two propositions are equivalent?

Solution:

The first method is to write a truth table for each proposition, and check that each row has the same truth value. The second is to use a chain of equivalences that starts at one proposition and ends at the other.

(c) To prove a chain of equivalences, there are many rules we can use (attached to the back of this handout). Fill in some of those rules here.

DeMorgan's Law:

Law of Implication:

Contrapositive:

Solution:

DeMorgan's Laws: $\neg(p \lor q) \equiv \neg p \land \neg q$, $\neg(p \land q) \equiv \neg p \lor \neg q$
Law of Implication: $p \to q \equiv \neg p \lor q$
Contrapositive: $p \to q \equiv \neg q \to \neg p$

(d) What is a predicate, a domain of discourse, and a quantifier?

Solution:

Predicate: A function, usually based on one or more variables, that is true or false.

Domain of Discourse: The universe of values that variables come from.

Quantifier: A claim about when the predicate is true. There are two quantifiers. $\forall$ says that the claim is true for all values, and $\exists$ says there exists a value for which the claim is true.

(e) When translating to predicate logic, how do you restrict to a smaller domain in a "for all"? How do you restrict to a smaller domain in an "exists"?

Solution:

If we need to restrict something quantified by a "for all", we use **implication**. If we need to restrict something quantifies by an "exists", we use **and**.

For example, suppose the domain of discourse is all animals. We translate "all birds can fly" to $\forall x(\text{Bird}(x) \rightarrow \text{Fly}(x))$. We translate "there is a bird that can fly" to $\exists x(\text{Bird}(x) \wedge \text{Fly}(x))$.

1. Translation: Running from my problems

Define a set of three atomic propositions, and use them to translate the following sentences.

(i) Whenever it's snowing and it's Friday, I am not going for a run.

(ii) I am going for a run because it is not snowing.

(iii) I am going for a run only if it is not Friday or not snowing.

Solution:

p: I am going for a run
q: It is snowing
r: It is Friday

(i) $(q \wedge r) \rightarrow \neg p$

(ii) $\neg q \rightarrow p$

(iii) $p \rightarrow (\neg r \vee \neg q)$

2. Truth Table

Draw a truth table for $(p \rightarrow \neg q) \rightarrow (r \oplus q)$

Solution:

p	q	r	$\neg q$	$p \rightarrow \neg q$	$r \oplus q$	$(p \rightarrow \neg q) \rightarrow (r \oplus q)$
T	T	T	F	F	F	T
T	T	F	F	F	T	T
T	F	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	T	F	F	T	T	T
F	F	T	T	T	T	T
F	F	F	T	T	F	F

3. Equivalences: Propositional Logic

Write a chain of logical equivalences to prove the following statements. Note that with propositional logic, you are expected to show all steps, including commutativity and associativity.

(a) $p \rightarrow q \equiv \neg(p \wedge \neg q)$

Solution:

These are equivalent. Below is the chain of equivalences.

$$\begin{aligned} p \rightarrow q &\equiv \neg p \vee q && \text{Law of Implication} \\ &\equiv \neg p \vee \neg\neg q && \text{Double Negation} \\ &\equiv \neg(p \wedge \neg q) && \text{DeMorgan's Law} \end{aligned}$$

(b) $((p \wedge q) \rightarrow r) \equiv (p \rightarrow r) \vee (q \rightarrow r)$

Solution:

$$
\begin{aligned}
(p \wedge q) \rightarrow r &\equiv \neg(p \wedge q) \vee r && \text{Law of Implication} \\
&\equiv (\neg p \vee \neg q) \vee r && \text{De Morgan's Law} \\
&\equiv (\neg p \vee \neg q) \vee (r \vee r) && \text{Idempotency} \\
&\equiv \neg p \vee (\neg q \vee (r \vee r)) && \text{Associativity} \\
&\equiv \neg p \vee ((\neg q \vee r) \vee r) && \text{Associativity} \\
&\equiv \neg p \vee (r \vee (\neg q \vee r)) && \text{Commutativity} \\
&\equiv \neg p \vee (r \vee (q \rightarrow r)) && \text{Law of Implication} \\
&\equiv (\neg p \vee r) \vee (q \rightarrow r) && \text{Associativity} \\
&\equiv (p \rightarrow r) \vee (q \rightarrow r) && \text{Law of Implication}
\end{aligned}
$$

4. Equivalences: Boolean Algebra

(a) Prove $p' + (p \cdot q) + (q' \cdot p) = 1$ via equivalences.

Solution:

$$
\begin{aligned}
p' + p \cdot q + q' \cdot p &\equiv p' + p \cdot q + p \cdot q' && \text{Commutativity} \\
&\equiv p' + p \cdot (q + q') && \text{Distributivity} \\
&\equiv p' + p \cdot 1 && \text{Complementarity} \\
&\equiv p' + p && \text{Identity} \\
&\equiv p + p' && \text{Commutativity} \\
&\equiv 1 && \text{Complementarity}
\end{aligned}
$$

5. Implications and Vacuous Truth

Alice and Bob's teacher says in class "if a number is prime, then the number is odd." Alice and Bob both believe that the teacher is wrong, but for different reasons.

(a) Alice says "9 is odd and not prime, so the implication is false." Is Alice's justification correct? Why or why not?

Solution:

No. She gave an example where the premise (number is prime) is false, and the conclusion is true. This doesn't disprove the claim!

(b) Bob says "2 is prime and not odd, so the implication is false." Is Bob's justification correct? Why or why not?

Solution:

Yes. He gave an example where the premise (number is prime) is true, and the conclusion is false. This does disprove the claim!

(c) Recall that this is the truth table for implications. Which row does Alice's example correspond to? Which row does Bob's example correspond to?

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Solution:

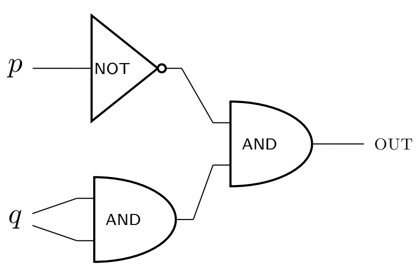
Alice's example corresponds to the row where p is false and q is true. Bob's example corresponds to the row where p is true and q is false.

(d) Observe that in order to show that $p \to q$ is false, you need an example where p is true and q is false. Examples where p is false don't disprove the implication! (Nothing to write for this part).

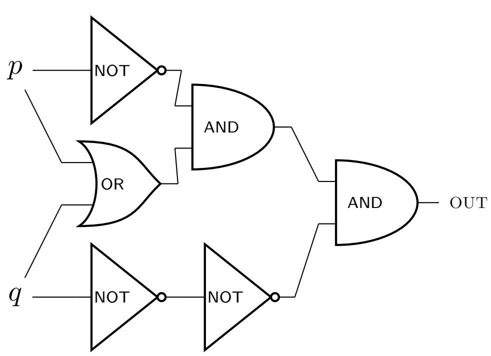
6. Circuits

Convert the following ciruits into logical expressions.

(i)



(ii)



Solution:

(i) $\neg p \wedge (q \wedge q)$

(ii) $((\neg p) \wedge (p \vee q)) \wedge \neg\neg q$

7. Predicate Logic: Logic to English

Let the domain of discourse be all animals. Let $\texttt{Panda}(x)$ ::= "x is a panda" and $\texttt{KungFu}(x)$ ::= x knows kung fu. Translate the following statements to English.

(a) $\exists x(\neg\, \texttt{Panda}(x) \wedge \texttt{KungFu}(x))$

Solution:

There exists a non-panda that knows kung fu.

(b) $\forall x(\texttt{Panda}(x) \rightarrow \texttt{KungFu}(x))$

Solution:

All pandas know kung fu.

(c) $\neg\exists y(\texttt{Panda}(y) \wedge \neg\, \texttt{KungFu}(y))$

Solution:

There does not exist a panda that doesn't know kung fu. (Same meaning as the statement in part b!)

Your friend translated the sentence "there exists a panda who knows kung fu" to $\exists x(\text{Panda}(x) \rightarrow \text{KungFu}(x))$. This is wrong! Let's understand why.

(d) Use the Law of Implications to rewrite the translation without the $\rightarrow$.

Solution:

$\exists x(\neg\text{Panda}(x) \vee \text{KungFu}(x))$

(e) Translate the predicate from (d) back to English. How does this differ from the intended meaning?

Solution:

Translation: There exists an animal that is not a panda or that knows kung fu.

The difference: If there was even one non-panda animal in the universe (e.g. a turtle), this condition would be satisfied. Similarly, if there was even one animal in the universe that knew kung fu, this condition would be satisfied. So, this expression has a very different meaning than "there exists a panda who knows kung fu".

(f) This is a warning to be very careful when including an implication nested under an exists! It should almost always be avoided, unless there is a forall involved as well. (Nothing to write for this part).

8. Predicate Logic: English to Logic

Express each of these system specifications using predicates, quantifiers, and logical connectives. For some of these problems, more than one translation will be reasonable depending on your choice of predicates. Be sure to define a domain!

(a) Everyone in this room loves logic.

Solution:

Let the domain be people in this room. Let $\texttt{LovesLogic}(x)$ be "x loves logic".

$$\forall x\, \texttt{LovesLogic}(x)$$

(b) Each name tag belongs to a student.

Solution:

Let the domain be name tags and students. Let $\texttt{NameTag}(x)$ be "x is a name tag", $\texttt{Student}(y)$ be "y is a student", and `Belongs(x, y)` be "x belongs to y".

$$\forall x(\texttt{NameTag}(x) \to \exists y(\texttt{Student}(y) \land \texttt{Belongs}(x, y))$$

(c) Each name tag belongs to exactly one student.
Think about what has to change about your solution to the previous part!

Solution:

Using the same definitions from part (b):

$$\forall x(\texttt{NameTag}(x) \to \exists y(\texttt{Student}(y) \land \texttt{Belongs}(x, y) \land \forall z((\texttt{Student}(z) \land (z \neq y)) \to \neg Belongs(x, z))))$$

(d) This classroom is unlocked only if all of the other classrooms on this floor are unlocked

Solution:

Let the domain be classrooms. Let p be the proposition "this classroom is unlocked". Let $\texttt{Unlocked}(x)$ be "x is unlocked" and $\texttt{SameFloor}(x)$ be "x is on this floor".

$$\forall x.((p \land \texttt{SameFloor}(x)) \to \texttt{Unlocked}(x))$$