CSE 390Z: Mathematics for Computation Workshop

Week 7 Workshop

0. Complete the Induction Proof

Consider the function $f(n)$ defined for integers $n \geq 1$ as follows:
$f(1) = 1$ for $n = 1$
$f(2) = 4$ for $n = 2$
$f(3) = 9$ for $n = 3$
$f(n) = f(n-1) - f(n-2) + f(n-3) + 2(2n-3)$ for $n \geq 4$

Prove by strong induction that for all $n \geq 1$, $f(n) = n^2$.

Complete the induction proof below:

Let P(n) be defined as ____________________. We will prove $P(n)$ is true for all integers $n \geq$ _____ by strong induction.

Base Cases:

So the base cases hold.

Inductive Hypothesis: Suppose for some arbitrary integer $k \geq$_____, P(j) is true for $1 \leq j \leq k$.

Inductive Step:

Goal: Show $P(k+1)$, i.e. show that $f(k+1) = (k+1)^2$.

$$\begin{aligned} f(k+1) &= f(k+1-1) - f(k+1-2) + f(k+1-3) + 2(2(k+1)-3) && \text{Definition of f} \\ &= f(k) - f(k-1) + f(k-2) + 2(2k-1) && \text{Algebra} \\ &= \end{aligned}$$

So P($k+1$) holds.

Conclusion: So by strong induction, P(n) is true for all integers $n \geq$_____.

1. Induction: Another Inequality

Prove by induction on n that for all integers $n \geq 4$ the inequality $n! > 2^n$ is true.

2. Induction: Divides

Prove that $9 \mid (n^3 + (n+1)^3 + (n+2)^3)$ for all $n > 1$ by induction.

3. Strong Induction: Stamp Collection

A store sells 3 cent and 5 cent stamps. Use strong induction to prove that you can make exactly n cents worth of stamps for all $n \geq 10$.

Hint: you'll need multiple base cases for this - think about how many steps back you need to go for your inductive step.

4. Strong Induction: Functions

Let a function f be defined by:

- $f(1) = 0$
- $f(2) = 12$
- $f(n) = 4 \cdot f(n-1) - 3 \cdot f(n-2)$ for $n \geq 3$

Prove that $f(n) = 2 \cdot 3^n - 6$ for any positive integer n.

5. Strong Induction: Collecting Candy

A store sells candy in packs of 4 and packs of 7. Let P(n) be defined as "You are able to buy n packs of candy". For example, $P(3)$ is not true, because you cannot buy exactly 3 packs of candy from the store. However, it turns out that P(n) is true for any $n \geq 18$. Use strong induction on n to prove this.

Hint: you'll need multiple base cases for this - think about how many steps back you need to go for your inductive step.

6. Structural Induction: a's and b's

Define a set $\mathcal{S}$ of character strings over the alphabet $\{a, b\}$ by:

- a and ab are in $\mathcal{S}$
- If $x \in \mathcal{S}$ and $y \in \mathcal{S}$, then $axb \in \mathcal{S}$ and $xy \in \mathcal{S}$

Prove by induction that every string in $\mathcal{S}$ has at least as many a's as it does b's.

7. Structural Induction: Divisible by 4

Define a set $\mathfrak{B}$ of numbers by:

- 4 and 12 are in $\mathfrak{B}$
- If $x \in \mathfrak{B}$ and $y \in \mathfrak{B}$, then $x + y \in \mathfrak{B}$ and $x - y \in \mathfrak{B}$

Prove by induction that every number in $\mathfrak{B}$ is divisible by 4.