CSE 390Z: Mathematics for Computation Workshop

Practice 311 Midterm Solutions

Name: ______________________________

UW ID: ______________________________

Instructions:

- This is a **simulated practice midterm**. You will be graded on your effort, not correctness, on this exam.
- Nevertheless, please treat this as if it is a real exam. That means that you should not discuss with your neighbors or use your devices during the next hour.
- If you find yourself needing to look concepts up in your notes or lecture materials, feel free to do so. Consider taking note of this so you can include it on your note sheet for the real exam, where you will not be able to have unlimited access to your notes.
- If you get stuck on a problem, consider moving on and coming back later. In the actual exam, there will likely be opportunity for partial credit.
- There are 4 problems on this exam, totaling 64 points.

1. Predicate Translation [20 points]

Let the domain of discourse be novels, comic books, movies, and TV shows. Assume the following predicates have been defined:

Novel(x) := x is a novel
Comic(x) := x is a comic book
Movie(x) := x is a movie
Show(x) := x is a TV show
Adaptation(x, y) := x is an adaptation of y

For parts (a) - (c), translate the English sentences to predicate logic.

(a) (4 points) A novel cannot be adapted into both a movie and a TV show.

Solution:

$\forall x(\text{Novel}(x) \rightarrow \forall m \forall s((\text{Movie}(m) \wedge \text{Show}(s)) \rightarrow \neg(\text{Adaptation}(m, x) \wedge \text{Adaptation}(s, x)))$

(b) (4 points) Every movie is an adaptation of a novel or a comic book.

Solution:

$\forall m(\text{Movie}(m) \rightarrow \exists x(\text{Adaptation}(m, x) \wedge (\text{Novel}(x) \vee \text{Comic}(x))))$

(c) (4 points) Every novel has been adapted into exactly one movie.

Solution:

$\forall x(\text{Novel}(x) \rightarrow \exists m(\text{Movie}(m) \wedge \text{Adaptation}(m, x) \wedge \forall n((\text{Movie}(n) \wedge (n \neq m)) \rightarrow \neg\text{Adaptation}(n, x))))$

OR

$\forall x(\text{Novel}(x) \rightarrow \exists m(\text{Movie}(m) \wedge \text{Adaptation}(m, x) \wedge \forall n(\text{Adaptation}(n, x) \rightarrow (\neg\text{Movie}(n) \vee n = m))))$

OR

$\forall x(\text{Novel}(x) \rightarrow \exists m(\text{Movie}(m) \wedge \text{Adaptation}(m, x) \wedge \forall n((\text{Adaptation}(n, x) \wedge \text{Movie}(n)) \rightarrow (n = m))))$

*Note that a great exercise is to show that the above 3 solutions are all logically equivalent :)

(d) (4 points) Translate the following statement to English:
$\exists x \exists y \exists z(\text{Novel}(x) \wedge \text{Movie}(y) \wedge \text{Movie}(z) \wedge (y \neq z) \wedge \text{Adaptation}(y, x) \wedge \text{Adaptation}(z, x))$

Solution:

There exists a novel that has been adapted into two different movies.

(e) (4 points) What is the negation of the statement from part (d), in English?
You do not need to show work for this part.

Solution:

No novel has been adapted into two different movies.

2. Sets [12 points]

Suppose that for sets A, B, C, the facts $A \subseteq B$ and $B \subseteq C$ are given. Write an English proof to show that $B \times A \subseteq C \times C$.

Solution:

Suppose that for sets A, B, C, we have $A \subseteq B$ and $B \subseteq C$ (these are our givens). Let $x \in B \times A$ be arbitrary. Then by definition of Cartesian Product, $x = (y, z)$ for $y \in B$ and $z \in A$. Then since $y \in B$ and $B \subseteq C$, $y \in C$. Similarly since $z \in A$ and $A \subseteq B$, $z \in B$. Then since $z \in B$ and $B \subseteq C$, we have $z \in C$. Therefore we have shown that $y \in C$ and $z \in C$. Then by definition of Cartesian Product, $x \in C \times C$. Since x was arbitrary, we have shown $B \times A \subseteq C \times C$.

3. Number Theory [12 points]

Recall this definition of even: Even$(x) := \exists y(x = 2y)$. Write an English proof using **proof by contradiction** to show that for all integers a, b, if $4 \mid (a^2 + b^2)$, then a and b are not both odd.

Solution:

Let a, b be arbitrary integers and suppose that $4 \mid (a^2 + b^2)$. Suppose for the sake of contradiction that a and b are both odd.

By definition of odd, $a = 2k + 1$ and $b = 2j + 1$ for some integers k, j.

By algebra,

$$a^2 + b^2 = (2k + 1)^2 + (2j + 1)^2 = 4k^2 + 4k + 1 + 4j^2 + 4j + 1 = 4(k^2 + k + j^2 + j) + 2$$

Since integers are closed under addition and multiplication, $k^2 + k + j^2 + j$ is an integer. In other words, $a^2 + b^2$ is 4 times an integer plus 2, which cannot be divisible by 4. This is a contradiction, since we stated that $4 \mid (a^2 + b^2)$.

Therefore, the original statement is true and both a, b cannot both be odd.

4. Induction [20 points]

Prove by induction that $(1+\pi)^n > 1 + n\pi$ for all integers $n \geq 2$.

Solution:

1. Let P(n) be the statement "$(1+\pi)^n > 1+n\pi$". We prove P(n) for all integers $n \geq 2$ by induction.

2. Base Case: When $n = 2$, the LHS is $(1+\pi)^2 = 1 + 2\pi + \pi^2$. The RHS is $1 + 2\pi$. Since $\pi^2 > 0$, $1 + 2\pi + \pi^2 > 1 + 2\pi$, so the Base Case holds.

3. Inductive Hypothesis: Suppose that P(k) holds for some arbitrary integer $k \geq 2$. Then $(1+\pi)^k > 1 + k\pi$.

4. Inductive Step:

$$\boxed{\text{Goal: Show } P(k+1)\text{, i.e. show } (1+\pi)^{k+1} > 1 + (k+1)\pi}$$

$$\begin{aligned}
(1+\pi)^{k+1} &= (1+\pi)(1+\pi)^k && \text{Definition of Exponent} \\
&> (1+\pi)(1+k\pi) && \text{By IH} \\
&= 1 + \pi + k\pi + k\pi^2 && \text{Algebra} \\
&= 1 + (k+1)\pi + k\pi^2 && \text{Algebra} \\
&> 1 + (k+1)\pi && \text{Since } k\pi^2 > 0
\end{aligned}$$

Thus $(1+\pi)^{k+1} > 1 + (k+1)\pi$. So P($k+1$) holds.

5. Thus we have proven P(n) for all integers $n \geq 2$ by induction.