CSE 390Z: Mathematics for Computation Workshop

Practice 311 Final Solutions

Name: ______________________________

UW ID: ______________________________

Instructions:

- This is a **simulated practice final**.You will be graded on your effort, not correctness, on this exam.
- Nevertheless, please treat this as if it is a real exam. That means that you should not discuss with your neighbors or use your devices during the next hour.
- If you find yourself needing to look concepts up in your notes or lecture materials, feel free to do so. Consider taking note of this so you can include it on your note sheet for the real exam, where you will not be able to have unlimited access to your notes.
- If you get stuck on a problem, consider moving on and coming back later. In the actual exam, there will likely be opportunity for partial credit.
- There are 8 problems on this exam, totaling 100 points.

Question	**Max points**
Translations	12
Language Representations	15
Induction I	15
Induction II	20
Modular Arithmetic	10
Relations	7
Irregularity	20
Grading Morale	1
Total	**100**

1. Translations [12 points]

For this problem, let the domain of discourse be all days of the year 2023 and all people. For each part, translate the English sentence to predicate logic. You may use the following predicates:
Person(x) := x is a person
Birthday(x,y) := x is y's birthday
Holiday(x) := x is a holiday
Weekend(x) := x is on the weekend

(a) [3 points] Everyone has a birthday.

Solution:

$$\forall x \exists y (\text{Person}(x) \rightarrow \text{Birthday}(y, x))$$

(b) [3 points] No one can have more than one birthday. (Note that a person could have zero birthdays.)

Solution:

$$\forall x (\text{Person}(x) \rightarrow \neg(\exists y \exists z (y \neq z \wedge \text{Birthday}(y, x) \wedge \text{Birthday}(z, x)))$$

(c) [3 points] Every holiday is also someone's birthday. (Note that the "someone" can be a different person for each holiday).

Solution:

$$\forall x (\text{Holiday}(x) \rightarrow \exists y (\text{Person}(y) \wedge \text{Birthday}(x, y))$$

(d) [3 points] Not all holidays are on the weekend.

Solution:

$$\neg \forall x (\text{Holiday}(x) \rightarrow \text{Weekend}(x))$$

2. Language Representations [15 points]

Let the alphabet be $\Sigma = \{a, b\}$.
Consider the language $L = \{w \in \Sigma^* : w$ where every occurence of a is not followed by $bb\}$.

Some strings in L include ϵ, a, ab, $aabaaab$, bbb, and $bbbbababa$.
Some strings **not** in L include abb, $bababb$, $aaabbaa$

(a) (5 points) Give a regular expression that represents L.

Solution:

$$b^*(a \cup ab)^*$$

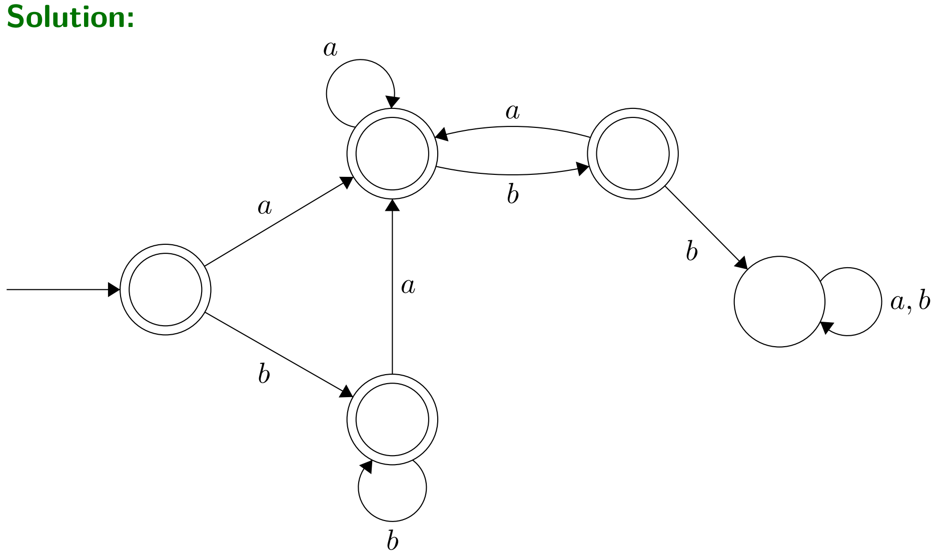
(b) (5 points) Give a CFG that represents L.

Solution:

S $\rightarrow$**BA**
B $\rightarrow b$**B** $\mid \epsilon$
A $\rightarrow$a**A** $\mid$ ab**A** $\mid \epsilon$

(c) (5 points) Give a DFA that represents L.

Solution:



3. Induction I [15 points]

Consider the following recursive definition of a_n:

$$a_1 = 1$$
$$a_2 = 1$$
$$a_n = \frac{1}{2}(a_{n-1} + \frac{2}{a_{n-2}}) \qquad \text{for } n > 2$$

Prove that $1 \le a_n \le 2$ for all integers $n \ge 1$.
Hint: To prove $1 \le x \le 2$ for some x, you should prove two facts: $1 \le x$ and $x \le 2$.

Solution:

Define P(n) to be $1 \le a_n \le 2$. We prove P(n) holds for all integers $n \ge 1$ by strong induction.

Base Case P(1), P(2) Observe that $a_1 = a_2 = 1$, and $1 \le 1 \le 2$. So P(1) and P(2) hold.
Inductive Hypothesis: Suppose that P(j) is true for all $1 \le j \le k$ for some arbitrary integer $k \ge 2$.
Inductive Step:

$$\begin{aligned}
a_{k+1} &= \frac{1}{2}(a_k + \frac{2}{a_{k-1}}) \\
&= \frac{a_k}{2} + \frac{1}{a_{k-1}} \\
&\le \frac{2}{2} + \frac{1}{a_{k-1}} && \text{By IH, since } a_k \le 2 \\
&\le 1 + \frac{1}{1} && \text{By IH, since } a_{k-1} \ge 1 \text{, so } \frac{1}{a_{k-1}} \le \frac{1}{1} \\
&= 2
\end{aligned}$$

$$\begin{aligned}
a_{k+1} &= \frac{1}{2}(a_k + \frac{2}{a_{k-1}}) \\
&= \frac{a_k}{2} + \frac{1}{a_{k-1}} \\
&\ge \frac{1}{2} + \frac{1}{a_{k-1}} && \text{By IH, since } a_k \ge 1 \\
&\ge \frac{1}{2} + \frac{1}{2} && \text{By IH, since } a_{k-1} \le 2 \text{, so } \frac{1}{a_{k-1}} \ge \frac{1}{2} \\
&= 1
\end{aligned}$$

So $1 \le a_{k+1} \le 2$.
Conclusion: Thus we have proven P(n) for all integers $n \ge 1$ by strong induction.

4. Induction II [20 points]

Recall the recursive definition of a list of integers:

- [ ] is the empty list
- If L is a list and a is an integer, then $a :: L$ is a list whose first element is a, followed by the elements of L.

Consider the following functions defined on lists:
len([ ]) $= 0$
len($x :: L$) $= 1+$ len(L)

inc([ ]) = [ ]
inc($x :: L$) $= (x+1) ::$ inc(L)

sum([ ]) $= 0$
sum($x :: L$) $= x +$ sum(L)

Prove that for all lists L, sum(inc(L)) = sum(L) + len(L).

Solution:
Let P(L) be "sum(inc(L)) = sum(L) + len(L)". We prove that P(L) is true for all lists L by structural induction.
Base Case: $L = [\,]$. Then:

$$\begin{aligned} \text{sum}(\text{inc}([\,])) &= \text{sum}([\,]) && \text{Definition of inc} \\ &= 0 && \text{Definition of sum} \\ &= 0 + 0 && \text{Algebra} \\ &= \text{sum}([\,]) + \text{len}([\,]) && \text{Definition of sum, len} \end{aligned}$$

Let M be an arbitrary list not covered by the base case. By the exclusion rule, M must be equal to $(x :: L)$ for some list L and element x.
Inductive Hypothesis: Suppose that P(L) is true.
Inductive Step: We aim to show that P(M) holds.

$$\begin{aligned} \text{sum}(\text{inc}(M)) &= \text{sum}(\text{inc}(x :: L)) && \text{Definition of M} \\ &= \text{sum}((x+1) :: \text{inc}(L)) && \text{Definition of inc} \\ &= (x+1) + \text{sum}(\text{inc}(L)) && \text{Definition of sum} \\ &= (x+1) + \text{sum}(L) + \text{len}(L) && \text{Inductive Hypothesis} \\ &= x + \text{sum}(L) + 1 + \text{len}(L) && \text{Algebra} \\ &= \text{sum}(x :: L) + \text{len}(x :: L) && \text{Definition of sum, len} \\ &= \text{sum}(M) + \text{len}(M) && \text{Definition of M} \end{aligned}$$

So P($x :: L$) holds.

Conclusion: Thus P(L) holds for all lists L by structural induction.

5. Modular Arithmetic [10 points]

(a) Prove or disprove: If $a \equiv b \pmod{10}$, then $a \equiv b \pmod{5}$. [5 points]

Solution:

True. Suppose that $a \equiv b \pmod{10}$. Then $10 \mid (a-b)$. Then there exists some integer k such that $a-b = 10k$ for some integer k. In particular, $a - b = 5(2k)$. Since integers are closed under multiplication, $2k$ must be an integer. Then $5 \mid (a-b)$. So $a \equiv b \pmod{5}$.

(b) Prove or disprove: If $a \equiv b \pmod{10}$, then $a \equiv b \pmod{20}$. [5 points]

Solution:

False. For example, for $a = 1$ and $b = 11$. Then $a \equiv b \pmod{10}$, but $a \not\equiv b \pmod{20}$.

6. Relations [7 points]

We define the relation R as $\{(x, y) : 3|(x-y)\}$.
List 3 elements of R

-
-
-

Solution:

- (3,3)
- (3,9)
- (6,0)

List the properties that R has out of the following: reflexive, transitive, symmetric, antisymmetric: (You do not need to show work for this part, just list the properties.)

Solution:

Reflexive, transitive, symmetric

7. Irregularity [20 points]

Prove that the set of strings $\{0^n 1 0^n : n \geq 0\}$ is not regular.

Solution:

$L = \{0^n 1 0^n \ : \ n \geq 0\}$. Let D be an arbitrary DFA, and suppose for contradiction that D accepts L. Consider $S = \{0^n : n \geq 0\}$. Since S contains infinitely many strings and D has a finite number of states, two strings in S must end up in the same state. Say these strings are 0^i and 0^j for some $i, j \geq 0$ such that $i \neq j$. Append the string 10^i to both of these strings. The two resulting strings are:

$a = 0^i 1 0^i$ Note that $a \in L$.

$b = 0^j 1 0^i$ Note that $b \notin L$, since $i \neq j$.

Since a and b end up in the same state, but $a \in L$ and $b \notin L$, that state must be both an accept and reject state, which is a contradiction. Since D was arbitrary, there is no DFA that recognizes L, so L is not regular.