CSE 390Z: Mathematics for Computation Workshop

QuickCheck: Structural Induction Solutions

Please submit a response to the following questions on Gradescope. We do not grade on accuracy, so please submit your best attempt. You may either typeset your responses or hand-write them. Note that hand-written solutions must be legible to be graded.

We have created **this template** if you choose to typeset with Latex. **This guide** has specific information about scanning and uploading pdf files to Gradescope.

0. How Many Ones?

The set T is defined as follows:

- Base case: $\epsilon \in T$
- Recursive Rules:
  If $x \in T$, then $11x \in T$
  If $x \in T$ and $y \in T$, then $x0y \in T$

Given the following recursively defined function

- numOnes(ϵ) $= 0$
- numOnes($11x$) $= 2 +$ numOnes(x)
- numOnes($x0y$) $=$ numOnes(x) $+$ numOnes(y)

Prove that for all strings n in T, numOnes(n) is even

Hint: In structural induction, the structure of your induction mirrors the recursive definition.

Solution:

Let $P(n)$ be "$2 \mid$ numOnes(n)". We will show that $P(n)$ is true for all $n \in T$ by structural induction.

Base Case ($n = \epsilon$):
numOnes(ϵ) $= 0$ definition of numOnes
$0 = 2 \cdot 0$ and $2|0$ by definition of divides.
Therefore P(ϵ) holds true.

Let s be an arbitrary element of T not covered by the base cases. By the exclusion rule, $s = 11x$ or $s = x0y$ for some elements $x, y \in T$.
Induction Hypothesis: Suppose $P(x)$ and $P(y)$ are true.

Induction Step:

Case 1: $11x$

numOnes($11x$) $= 2 +$ numOnes(x) by definition of numOnes. By the inductive hypothesis, $2 \mid$ numOnes(x). Therefore, by definition of divides numOnes(x) $= 2z$ for some integer z. Thus,

$$\text{numOnes}(11x) = 2 + \text{numOnes}(x) = 2z + 2 = 2(z + 1)$$

Therefore, by definition of divides, $2 \mid$ numOnes($11x$). Therefore, P($11x$) holds.

Case 2: $x0y$

numOnes($x0y$) = numOnes(x) + numOnes(y) by definition of numOnes. By the induction hypothesis, 2 | numOnes(x) and 2 | numOnes(y). Therefore, by definition of divides, numOnes(x) = 2z for some integer z and numOnes(y) = 2q for some integer q. Thus,

$$\text{numOnes}(x0y) = \text{numOnes}(x) + \text{numOnes}(y) = 2z + 2q = 2(z + q)$$

Therefore, by definition of divides 2 | numOnes($x0y$). Therefore, P($x0y$) holds.

The result follows for all $n \in T$ by structural induction.

1. Video Solution

Watch **this video** on the solution **after** making an initial attempt. Then, answer the following questions.

(a) What is one thing you took away from the video solution?

(b) What topic from the quick check or lecture would you most like to review in workshop?