CSE 390z: Mathematics for Computation Workshop

QuickCheck: Induction Proofs Solutions

Please submit a response to the following questions on Gradescope. We do not grade on accuracy, so please submit your best attempt. You may either typeset your responses or hand-write them. Note that hand-written solutions must be legible to be graded.

We have created **this template** if you choose to typeset with Latex. **This guide** has specific information about scanning and uploading pdf files to Gradescope.

0. Weak Induction, Strong Induction, ...What's The Difference?

Recall that the nth Fibonacci Number is given by:

$$f(n) = f(n-1) + f(n-2)$$

for all natural numbers $n \geq 2$, where $f(0) = 0, f(1) = 1$.

Prove that $f(n) \leq 2^n$ for all $n \in \mathbb{N}$.

(a) Prove this using *weak induction*.

Note: You may find the following observation useful for the inductive step:

$$f(n+1) = f(n) + f(n-1) \leq f(n) + f(n) = 2 \cdot f(n)$$

Solution:

1. Let $P(n)$ be "$f(n) \leq 2^n$". We will prove $P(n)$ is true for all $n \in \mathbb{N}$, by induction.

2. Base cases (n = 0, n = 1):

$f(0) = 0$ and $2^0 = 1$, since $0 \leq 1$, $P(0)$ is true.
$f(1) = 1$ and $2^1 = 2$, since $1 \leq 2$, $P(1)$ is true.

3. Inductive Hypothesis: Suppose that $P(k)$ is true for some arbitrary natural number $k \geq 1$.

4. Inductive Step:

$$\boxed{\text{Goal: Show } P(k+1) \text{, i.e. show } f(k+1) \leq 2^{k+1}}$$

$$\begin{aligned}
f(k+1) &= f(k) + f(k-1) && \text{(Def. of Fibonacci)} \\
&\leq f(k) + f(k) && \text{(Since } f(k-1) \leq f(k)\text{)} \\
&= 2 \cdot f(k) && \text{(Factor)} \\
&\leq 2 \cdot 2^k && \text{(I.H.)} \\
&= 2^{k+1} && \text{(Add Exponents)}
\end{aligned}$$

5. So by induction, $P(n)$ is true for all $n \in \mathbb{N}$.

(b) Prove this using *strong induction*.

Solution:

1. Let $P(n)$ be "$f(n) \leq 2^n$". We will prove $P(n)$ is true for all $n \in \mathbb{N}$, by *strong* induction.

2. Base cases (n = 0, n = 1):

$$f(0) = 0 \text{ and } 2^0 = 1, \text{ since } 0 \leq 1,\ P(0) \text{ is true.}$$
$$f(1) = 1 \text{ and } 2^1 = 2, \text{ since } 1 \leq 2,\ P(1) \text{ is true.}$$

3. Inductive Hypothesis: Suppose that $P(j)$ is true for *all natural numbers* $0 \leq j \leq k$ for some arbitrary natural number $k \geq 1$.

4. Inductive Step:

$$\boxed{\text{Goal: Show } P(k+1), \text{ i.e. show } f(k+1) \leq 2^{k+1}}$$

$$\begin{aligned}
f(k+1) &= f(k) + f(k-1) && \text{(Def. of Fibonacci)}\\
&\leq 2^k + f(k-1) && \text{(I.H.)}\\
&\leq 2^k + 2^{k-1} && \text{(I.H.)}\\
&\leq 2^k + 2^k && (k \geq 1)\\
&= 2 \cdot 2^k && \text{(Factor)}\\
&= 2^{k+1} && \text{(Add Exponents)}
\end{aligned}$$

5. So by *strong* induction, $P(n)$ is true for all $n \in \mathbb{N}$.

(c) How are the weak and strong induction proofs the same? What parts are different?

Solution:

The approaches are *nearly identical* besides for the inductive hypothesis and inductive step. Hopefully, it is now clear that strong induction gives more evidence (you assume more things hold true in the I.H.) than weak induction. This means that the *strong* induction solution to this problem is a little more natural, since we are dealing with the two previous terms in the Fibonacci sequence. However, with a clever trick, weak induction is a simple solution as well. It will generally be the case that weak or strong induction is easier for a given problem, always choose the easiest of the two if they're both possible. Lastly, remember that strong induction will always work on a problem that can be solved by weak induction, but not necessarily the other way around.

1. Video Solution

Watch **this** solution video **after** making an initial attempt. Then, answer the following questions.

(a) What is one thing you took away from the video solution?

(b) What topic from the quick check or lecture would you most like to review in workshop?