CSE 390Z: Mathematics of Computing

Week 9 Workshop Solutions

Conceptual Review

Relations definitions: Let R be a relation on A. In other words, $R \subseteq A \times A$. Then:

- R is reflexive iff for all $a \in A$, $(a, a) \in R$.
- R is symmetric iff for all a, b, if $(a, b) \in R$, then $(b, a) \in R$.
- R is antisymmetric iff for all a, b, if $(a, b) \in R$ and $a \neq b$, then $(b, a) \notin R$.
- R is transitive iff for all a, b, if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

Let R, S be relations on A. Then:

- $R \circ S = \{(a, c) : \exists b \text{ such that } (a, b) \in R \text{ and } (b, c) \in S\}$

1. Relations Examples

(a) Suppose that R, S are relations on the integers, where $R = \{(1,2), (4,3), (5,5)\}$ and $S = \{(2,5), (2,7), (3,3)\}$. What is $R \circ S$? What is $S \circ R$?

Solution:

$R \circ S = \{(1,5), (1,7), (4,3)\}$
$S \circ R = \{(2,5)\}$

(b) Consider the relation $R \subseteq \mathbb{Z} \times \mathbb{Z}$ defined by $(a, b) \in R$ iff $a \leq b + 1$. List 3 pairs of integers that are in R, and 3 pairs of integers that are not.

Solution:

In R: $(0,0), (1,0), (-1,0)$
Not in R: $(2,0), (3,0), (17,5)$

(c) Consider the relation $R \subseteq \mathbb{Z} \times \mathbb{Z}$ defined by $(a, b) \in R$ iff $a \leq b+1$. Determine if R is reflexive, symmetric, antisymmetric, and/or transitive. If a relation has a property, explain why. If not, state a counterexample.

Solution:

- Reflexive: Yes. For any integer a, it is true that $a \leq a + 1$. So $(a, a) \in R$.
- Symmetric: No. For example, $(0, 20) \in R$ but $(20, 0) \notin R$.
- Antisymmetric: No. For example $(0, 1) \in R$ and $(1, 0) \in R$.
- Transitive: No. For example $(2, 1) \in R$ and $(1, 0) \in R$, but $(2, 0) \notin R$.

2. Relations Proofs

Suppose that $R, S \subseteq \mathbb{Z} \times \mathbb{Z}$ are relations.

(a) Prove or disprove: If R and S are transitive, $R \cup S$ is transitive.

Solution:

False. Let $R = \{(1,2)\}$, $S = \{(2,1)\}$. By definition, R and S are transitive. By definition of intersect, $R \cup S = \{(1,2),(2,1)\}$. However, if $R \cup S$ was transitive, we would require $(1,1)$ to be in $R \cup S$, because (1,2) and (2,1) is in $R \cup S$. However, this is not the case. Therefore the claim is false.

(b) Prove or disprove: If R and S are reflexive, then $R \circ S$ is reflexive.

Solution:

True. Let $a \in \mathbb{Z}$ be arbitrary. Then $(a,a) \in R$ and $(a,a) \in S$ by definition of reflexive. Then $(a,a) \in R \circ S$. So $R \circ S$ is reflexive.

(c) Prove or disprove: If $R \circ S$ is reflexive, then R and S are reflexive.

Solution:

False. Let $R = \{(a, a+1) : a \in \mathbb{Z}\}$. In other words, $R = \{...(-2,-1),(-1,0),(0,1),(1,2)...\}$. Let $S = \{(a, a-1) : a \in \mathbb{Z}\}$. In other words, $S = \{...(-1,-2),(0,-1),(1,0),(2,1)...\}$. Then for any arbitrary $a \in \mathbb{Z}$, we have $(a, a+1) \in R$ and $(a+1, a) \in S$. So $(a,a) \in R \circ S$. So $R \circ S$ is reflexive. Thus we have found an example where $R \circ S$ is reflexive, but R and S are not.

(d) Prove or disprove: If R is symmetric, $\overline{R}$ (the complement of R) is symmetric.

Solution:

True. Since R is symmetric, we know the following.

$$\forall a \forall b\ [(a,b) \in R \to (b,a) \in R]$$

Taking the contrapositive, this is equivalent to:

$$\forall a \forall b\ [(b,a) \notin R \to (a,b) \notin R]$$

By the definition of complement, this is equivalent to:

$$\forall a \forall b\ [(b,a) \in \overline{R} \to (a,b) \in \overline{R}]$$

This is the definition of $\overline{R}$ being symmetric.

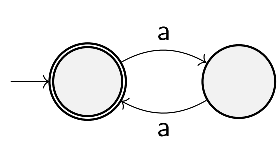
3. Constructing DFAs

For each of the following, construct a DFA for the specified language.

(a) Strings with an even number of a's ($\Sigma = \{a\}$).

Solution:

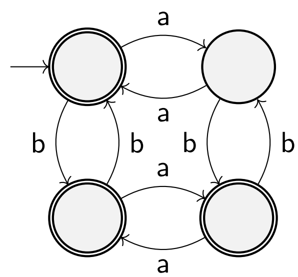
DFA:



(b) Strings with an even number of a's or an odd number of b's ($\Sigma = \{a, b\}$).

Solution:

DFA:



(c) Strings of a's and b's with odd length ($\Sigma = \{a, b\}$).

Solution:

DFA: