CSE 390Z: Mathematics of Computing

Week 8 Workshop Solutions

Conceptual Review

Space to take notes on Structural Induction, Regular Expressions, and CFGs:

1. Structural Induction x CFG Example

Consider the following CFG:

$$S \to SS \mid 0S1 \mid 1S0 \mid \epsilon$$

Prove that every string generated by this CFG has an equal number of 1's and 0's.

Hint: You may wish to define the functions $\#_0(x), \#_1(x)$ on a string x.

Solution:

First we observe that the language defined by this CFG can be represented by a recursively defined set. Define a set S as follows:
Basis Rule: $\epsilon \in S$
Recursive Rule: If $x, y \in S$, then $0x1, 1x0, xy \in S$.

Now we perform structural induction on the recursively defined set. Define the functions $\#_0(t), \#_1(t)$ to be the number of 0's and 1's respectively in the string t.

1 For a string t, let P(t) be defined as "$\#_0(t) = \#_1(t)$". We will prove P(t) is true for all strings $t \in S$ by structural induction.

2 **Base Case** ($t = \epsilon$): By definition, the empty string contains no characters, so $\#_0(t) = 0 = \#_1(t)$

3 **Inductive Hypothesis:** Suppose P(x), P(y) hold for some arbitrary strings x, y.

4 **Inductive Step:**
Case 1: Goal is to show P$(0x1)$ holds.
By the IH, $\#_0(x) = \#_1(x)$. Then observe that:

$$\#_0(0x1) = \#_0(x) + 1 = \#_1(x) + 1 = \#_1(0x1)$$

Therefore $\#_0(0x1) = \#_1(0x1)$. This proves P$(0x1)$.

Case 2: Goal is to show P$(1x0)$ holds.
By the IH, $\#_0(x) = \#_1(x)$. Then observe that:

$$\#_0(1x0) = \#_0(x) + 1 = \#_1(x) + 1 = \#_1(1x0)$$

Therefore $\#_0(1x0) = \#_1(1x0)$. This proves P$(1x0)$.

Case 3: Goal is to show P(xy) holds.
By the IH, $\#_0(x) = \#_1(x)$ and $\#_0(y) = \#_1(y)$. Then observe that:

$$\#_0(xy) = \#_0(x) + \#_0(y) = \#_1(x) + \#_1(y) = \#_1(xy)$$

Therefore $\#_0(xy) = \#_1(xy)$. This proves P(xy).

5 So by structural induction, P(t) is true for all strings $t \in S$.

2. Context Free Grammars

Consider the following CFG which generates strings from the language V $:= \{0, 1, 2, 3, 4\}^*$

$$\textbf{S} \rightarrow 0\textbf{X}4$$
$$\textbf{X} \rightarrow 1\textbf{X}3 \mid 2$$

List 5 strings generated by the CFG and 5 strings from V not generated by the CFG. Then, summarize this CFG in your own words.

Solution:

Accepted:

- 024
- 01234
- 0112334
- 011123334
- 01111233334

Rejected:

- ϵ
- 2
- 0244
- 011234
- 10234

This CFG is all strings of the form $0\ 1^m\ 2\ 3^m\ 4$, where $m \geq 0$. That is, it's all strings made of one 0, followed by zero or more 1's, followed by a 2, followed by the same number of 3's as 1's, followed by one 4.

3. Constructing Languages

For each of the following, construct a regular expression and a CFG for the specified language.

(a) Strings from the language $S := \{a\}^*$ with an even number of a's.

Solution:

RegEx: $(aa)^*$

CFG: $\textbf{S} \rightarrow aa\textbf{S}|\varepsilon$

(b) Strings from the language $S := \{a, b\}^*$ with odd length.

Solution:

$(aa \cup ab \cup ba \cup bb)^*(a \cup b)$
CFG:

$$\textbf{S} \rightarrow \textbf{CS}|a|b$$
$$\textbf{C} \rightarrow aa\textbf{C}|ab\textbf{C}|ba\textbf{C}|bb\textbf{C}|\varepsilon$$

(c) (Challenge) Strings from the language $S := \{a, b\}^*$ with an even number of a's or an odd number of b's.

Solution:

RegEx: $(b^*ab^*ab^*)^* \cup (a^* \cup a^*ba^*ba^*)^*b(a^* \cup a^*ba^*ba^*)^*$

CFG:

$$\begin{aligned}
\mathbf{S} &\to \mathbf{T}|\mathbf{R}\\
\mathbf{T} &\to \mathbf{B}a\mathbf{B}a\mathbf{B}\mathbf{T}|\varepsilon\\
\mathbf{R} &\to \mathbf{A}\mathbf{R}\mathbf{A}|\varepsilon\\
\mathbf{B} &\to b\mathbf{B}|\varepsilon\\
\mathbf{A} &\to a\mathbf{A}|a\mathbf{A}ba\mathbf{A}ba\mathbf{A}|ba\mathbf{A}ba\mathbf{A}|a\mathbf{A}ba\mathbf{A}b|a\mathbf{A}bba\mathbf{A}|a\mathbf{A}bb|b\mathbf{A}b|bb\mathbf{A}|bb
\end{aligned}$$

4. Structural Induction on Palindromes

Consider the following *recursive* defintion of the set B of palindrome binary strings:

- **Base case:** $\varepsilon \in B$, $0 \in B$, $1 \in B$.
- **Recursive steps**:
  - If $s \in B$, then $0s0 \in E$, $1s1 \in B$, and $ss \in B$.

  Now define the functions numOnes(x) and numZeros(x) to be the number of 1s and 0s respectively in the string x.

Use *structural induction* to prove that for any string $s \in B$, numOnes$(s) \cdot$ numZeros(s) is even.

Solution:

Proof. Define P(n) to be "$2 \mid$ numOnes$(s) \cdot$ numZeros(s)". We will show P(n) for all $n \in B$ by structural induction.

Base Cases:

- $s = \varepsilon$: numOnes$(\varepsilon) \cdot$ numZeros$(\varepsilon) = 0 \cdot 0 = 0 = 2 \cdot 0$, thus P$(\varepsilon)$ holds.
- $s = 0$: numOnes$(0) \cdot$ numZeros$(0) = 0 \cdot 1 = 0 = 2 \cdot 0$, thus P(0) holds.
- $s = 1$: numOnes$(1) \cdot$ numZeros$(1) = 1 \cdot 0 = 0 = 2 \cdot 0$, thus P(1) holds.

Inductive Hypothesis: Suppose P(s) holds for an arbitrary string $s \in B$. **Inductive Step:**

- **Case 1:** $0s0$

$$\begin{aligned}\text{numOnes}(0s0) \cdot \text{numZeros}(0s0) &= (2 + \text{numZeros}(s)) \cdot \text{numOnes}(s) && \text{(Def. of numZeros, numOnes)} \\ &= 2 \cdot \text{numOnes}(s) + \text{numZeros}(s) \cdot \text{numOnes}(s)\end{aligned}$$

  By the I.H., $2 \mid$ numZeros$(s) \cdot$numOnes(s), thus there is an integer k s.t. numZeros$(s) \cdot$numOnes$(s) = 2 \cdot k$. We can substitute this to get $2 \cdot$ numOnes$(s) + 2 \cdot k$, which we can rearrange to get $2 \cdot ($numOnes$(s) + k)$, thus $2 \mid$ numOnes$(0s0) \cdot$ numZeros$(0s0)$ and P$(0s0)$ holds.

- **Case 2:** $1s1$

$$\begin{aligned}\text{numOnes}(1s1) \cdot \text{numZeros}(1s1) &= \text{numZeros}(s) \cdot (2 + \text{numOnes}(s)) && \text{(Def. of numZeros, numOnes)} \\ &= 2 \cdot \text{numZeros}(s) + \text{numZeros}(s) \cdot \text{numOnes}(s)\end{aligned}$$

  By the I.H., $2 \mid$ numZeros$(s) \cdot$numOnes(s), thus there is an integer k s.t. numZeros$(s) \cdot$numOnes$(s) = 2 \cdot k$. We can substitute this to get $2 \cdot$ numZeros$(s) + 2 \cdot k$, which we can rearrange to get $2 \cdot ($numZeros$(s) + k)$, thus $2 \mid$ numOnes$(1s1) \cdot$ numZeros$(1s1)$ and P$(1s1)$ holds.

- **Case 3:** ss

$$\begin{aligned}\text{numOnes}(ss) \cdot \text{numZeros}(ss) &= (2 \cdot \text{numOnes}(s)) \cdot (2 \cdot \text{numZeros}(s)) && \text{(Def. of numZeros, numOnes)} \\ &= 4 \cdot \text{numOnes}(s) \cdot \text{numZeros}(s)\end{aligned}$$

  By the I.H., $2 \mid$ numZeros$(s) \cdot$numOnes(s), thus there is an integer k s.t. numZeros$(s) \cdot$numOnes$(s) = 2 \cdot k$. We can substitute this to get $4 \cdot 2 \cdot k = 2 \cdot (4 \cdot k)$, thus $2 \mid$ numOnes$(ss) \cdot$ numZeros(ss) and P(ss) holds.

Thus, P(s) holds for all $s \in B$ by structural induction. $\square$

5. Relations

Suppose A is nonempty set and $R, S \subset A \times A$. The universe that A exists in is only integers.

(a) Prove or disprove: If R and S are reflexive, $R \cap S$ is reflexive.

Solution:

True. Suppose R and S are reflexive relations. Let $a \in A$ be arbitrary. Since R is reflexive, $(a, a) \in R$. Since S is reflexive, $(a, a) \in S$. Then by definition of intersect, $(a, a) \in R \cap S$. Since a was arbitrary, by definition of reflexive, $R \cap S$ is reflexive.

(b) Prove or disprove: If R and S are transitive, $R \cup S$ is transitive.

Solution:

False. Let $A = \{1, 2\}$, $R = \{(1, 2)\}$, $S = \{(2, 1)\}$. By definition, R and S are transitive. By definition of intersect, $R \cup S = \{(1, 2), (2, 1)\}$. However, if $R \cup S$ was transitive I would expect $(1, 1)$ to be in $R \cup S$ because (1,2) and (2,1) is in $R \cup S$. However, this is not the case. Therefore the claim is false.

(c) Prove or disprove: If R is symmetric, $\overline{R}$ is symmetric.

Solution:

True. Since R is symmetric, we know the following.

$$\forall a, \forall b, (a, b) \in R \rightarrow (b, a) \in R$$

Taking the contrapositive, this is equivalent to:

$$\forall a, \forall b, (b, a) \notin R \rightarrow (a, b) \notin R$$

By the definition of complement, this is equivalent to:

$$\forall a, \forall b, (b, a) \in \overline{R} \rightarrow (a, b) \in \overline{R}$$

This is the definition of $\overline{R}$ being symmetric.