Conceptual Review
Space to take notes on Structural Induction, Regular Expressions, and CFGs:
1. Structural Induction x CFG Example
Consider the following CFG:

\[ S \rightarrow SS \mid 0S1 \mid 1S0 \mid \epsilon \]

Prove that every string generated by this CFG has an equal number of 1's and 0's.

**Hint:** You may wish to define the functions \( \#_0(x) \), \( \#_1(x) \) on a string \( x \).

**Solution:**
First we observe that the language defined by this CFG can be represented by a recursively defined set. Define a set \( S \) as follows:

**Basis Rule:** \( \epsilon \in S \)

**Recursive Rule:** If \( x, y \in S \), then \( 0x1, 1x0, xy \in S \).

Now we perform structural induction on the recursively defined set. Define the functions \( \#_0(t) \), \( \#_1(t) \) to be the number of 0's and 1's respectively in the string \( t \).

1. For a string \( t \), let \( P(t) \) be defined as "\( \#_0(t) = \#_1(t) \)". We will prove \( P(t) \) is true for all strings \( t \in S \) by structural induction.

2. **Base Case** \( (t = \epsilon) \): By definition, the empty string contains no characters, so \( \#_0(\epsilon) = 0 = \#_1(\epsilon) \)

3. **Inductive Hypothesis:** Suppose \( P(x), P(y) \) hold for some arbitrary strings \( x, y \).

4. **Inductive Step:**
   - **Case 1:** Goal is to show \( P(0x1) \) holds.
     By the IH, \( \#_0(x) = \#_1(x) \). Then observe that:
     \[
     \#_0(0x1) = \#_0(x) + 1 = \#_1(x) + 1 = \#_1(0x1)
     \]
     Therefore \( \#_0(0x1) = \#_1(0x1) \). This proves \( P(0x1) \).

   - **Case 2:** Goal is to show \( P(1x0) \) holds.
     By the IH, \( \#_0(x) = \#_1(x) \). Then observe that:
     \[
     \#_0(1x0) = \#_0(x) + 1 = \#_1(x) + 1 = \#_1(1x0)
     \]
     Therefore \( \#_0(1x0) = \#_1(1x0) \). This proves \( P(1x0) \).

   - **Case 3:** Goal is to show \( P(xy) \) holds.
     By the IH, \( \#_0(x) = \#_1(x) \) and \( \#_0(y) = \#_1(y) \). Then observe that:
     \[
     \#_0(xy) = \#_0(x) + \#_0(y) = \#_1(x) + \#_1(y) = \#_1(xy)
     \]
     Therefore \( \#_0(xy) = \#_1(xy) \). This proves \( P(xy) \).

5. So by structural induction, \( P(t) \) is true for all strings \( t \in S \).
2. Context Free Grammars
Consider the following CFG which generates strings from the language \( V := \{0, 1, 2, 3, 4\}^* \)

\[
\begin{align*}
S & \rightarrow 0X4 \\
X & \rightarrow 1X3 \mid 2
\end{align*}
\]

List 5 strings generated by the CFG and 5 strings from \( V \) not generated by the CFG. Then, summarize this CFG in your own words.

**Solution:**

**Accepted:**
- 024
- 01234
- 0112334
- 011123334
- 01111233334

**Rejected:**
- \( \epsilon \)
- 2
- 0244
- 011234
- 10234

This CFG is all strings of the form \( 0 \ 1^m \ 2 \ 3^m \ 4 \), where \( m \geq 0 \). That is, it’s all strings made of one 0, followed by zero or more 1’s, followed by a 2, followed by the same number of 3’s as 1’s, followed by one 4.

3. Constructing Languages
For each of the following, construct a regular expression and a CFG for the specified language.

(a) Strings from the language \( S := \{a\}^* \) with an even number of \( a \)'s.

**Solution:**

RegEx: \((aa)^*\)

CFG: \( S \rightarrow aaS | \epsilon \)

(b) Strings from the language \( S := \{a, b\}^* \) with odd length.

**Solution:**

\((aa \cup ab \cup ba \cup bb)^*(a \cup b)\)

CFG:

\[
\begin{align*}
S & \rightarrow CS|a|b \\
C & \rightarrow aaC|abC|baC|bbC|\epsilon
\end{align*}
\]

(c) (Challenge) Strings from the language \( S := \{a, b\}^* \) with an even number of \( a \)'s or an odd number of \( b \)'s.

**Solution:**

RegEx: \((b^*ab^*ab^*)^* \cup (a^* \cup a^*ba^*ba^*)^*b(a^* \cup a^*ba^*ba^*)^*\)
CFG:

\[ S \rightarrow T \, | \, R \]
\[ T \rightarrow B \, a \, B \, b \, T \, | \, \varepsilon \]
\[ R \rightarrow A \, R \, A \, | \, \varepsilon \]
\[ B \rightarrow b \, B \, | \, \varepsilon \]
\[ A \rightarrow a \, A \, | \, a \, A \, b \, a \, A \, b \, a \, A \, | \, b \, a \, A \, b \, a \, A \, | \, a \, A \, b \, b \, A \, b \, A \, b \, A \, | \, b \, b \, A \, | \, b \, b \]
4. Structural Induction on Palindromes

Consider the following recursive definition of the set $B$ of palindrome binary strings:

- **Base case:** $\varepsilon \in B$, $0 \in B$, $1 \in B$.
- **Recursive steps:**
  - If $s \in B$, then $0s0 \in E$, $1s1 \in B$, and $ss \in B$.

Now define the functions $\text{numOnes}(x)$ and $\text{numZeros}(x)$ to be the number of 1s and 0s respectively in the string $x$.

Use structural induction to prove that for any string $s \in B$, $\text{numOnes}(s) \cdot \text{numZeros}(s)$ is even.

**Solution:**

*Proof.* Define $P(n)$ to be "$2 \mid \text{numOnes}(s) \cdot \text{numZeros}(s)$". We will show $P(n)$ for all $n \in B$ by structural induction.

**Base Cases:**

- **Case 1:** $0s0$

  
  \[
  \text{numOnes}(0s0) \cdot \text{numZeros}(0s0) = (2 + \text{numZeros}(s)) \cdot \text{numOnes}(s) \quad \text{(Def. of numZeros, numOnes)}
  \]
  
  \[
  = 2 \cdot \text{numOnes}(s) + \text{numZeros}(s) \cdot \text{numOnes}(s)
  \]

  By the I.H., $2 \mid \text{numZeros}(s) \cdot \text{numOnes}(s)$, thus there is an integer $k$ s.t. $\text{numZeros}(s) \cdot \text{numOnes}(s) = 2 \cdot k$. We can substitute this to get $2 \cdot \text{numOnes}(s) + 2 \cdot k$, which we can rearrange to get $2 \cdot (\text{numOnes}(s) + k)$, thus $2 \mid \text{numOnes}(0s0) \cdot \text{numZeros}(0s0)$ and $P(0s0)$ holds.

- **Case 2:** $1s1$

  \[
  \text{numOnes}(1s1) \cdot \text{numZeros}(1s1) = \text{numZeros}(s) \cdot (2 + \text{numOnes}(s)) \quad \text{(Def. of numZeros, numOnes)}
  \]
  
  \[
  = 2 \cdot \text{numZeros}(s) + \text{numZeros}(s) \cdot \text{numOnes}(s)
  \]

  By the I.H., $2 \mid \text{numZeros}(s) \cdot \text{numOnes}(s)$, thus there is an integer $k$ s.t. $\text{numZeros}(s) \cdot \text{numOnes}(s) = 2 \cdot k$. We can substitute this to get $2 \cdot \text{numZeros}(s) + 2 \cdot k$, which we can rearrange to get $2 \cdot (\text{numZeros}(s) + k)$, thus $2 \mid \text{numOnes}(1s1) \cdot \text{numZeros}(1s1)$ and $P(1s1)$ holds.

- **Case 3:** $ss$

  \[
  \text{numOnes}(ss) \cdot \text{numZeros}(ss) = (2 \cdot \text{numOnes}(s)) \cdot (2 \cdot \text{numZeros}(s)) \quad \text{(Def. of numZeros, numOnes)}
  \]
  
  \[
  = 4 \cdot \text{numOnes}(s) \cdot \text{numZeros}(s)
  \]

  By the I.H., $2 \mid \text{numZeros}(s) \cdot \text{numOnes}(s)$, thus there is an integer $k$ s.t. $\text{numZeros}(s) \cdot \text{numOnes}(s) = 2 \cdot k$. We can substitute this to get $4 \cdot 2 \cdot k = 2 \cdot (4 \cdot k)$, thus $2 \mid \text{numOnes}(ss) \cdot \text{numZeros}(ss)$ and $P(ss)$ holds.

Thus, $P(s)$ holds for all $s \in B$ by structural induction.
5. Relations
Suppose $A$ is nonempty set and $R, S \subseteq A \times A$. The universe that $A$ exists in is only integers.

(a) Prove or disprove: If $R$ and $S$ are reflexive, $R \cap S$ is reflexive.

Solution:
True. Suppose $R$ and $S$ are reflexive relations. Let $a \in A$ be arbitrary. Since $R$ is reflexive, $(a, a) \in R$. Since $S$ is reflexive, $(a, a) \in S$. Then by definition of intersect, $(a, a) \in R \cap S$. Since $a$ was arbitrary, by definition of reflexive, $R \cap S$ is reflexive.

(b) Prove or disprove: If $R$ and $S$ are transitive, $R \cup S$ is transitive.

Solution:
False. Let $A = \{1, 2\}$, $R = \{(1, 2)\}$, $S = \{(2, 1)\}$. By definition, $R$ and $S$ are transitive. By definition of intersect, $R \cup S = \{(1, 2), (2, 1)\}$. However, if $R \cup S$ was transitive I would expect $(1, 1)$ to be in $R \cup S$ because $(1, 2)$ and $(2, 1)$ is in $R \cup S$. However, this is not the case. Therefore the claim is false.

(c) Prove or disprove: If $R$ is symmetric, $\overline{R}$ is symmetric.

Solution:
True. Since $R$ is symmetric, we know the following.

$$\forall a, \forall b, (a, b) \in R \rightarrow (b, a) \in R$$

Taking the contrapositive, this is equivalent to:

$$\forall a, \forall b, (b, a) \notin R \rightarrow (a, b) \notin R$$

By the definition of complement, this is equivalent to:

$$\forall a, \forall b, (b, a) \in \overline{R} \rightarrow (a, b) \in \overline{R}$$

This is the definition of $\overline{R}$ being symmetric.