CSE 390Z: Mathematics for Computation Workshop

Practice 311 Midterm Solutions

Name: ______________________________

UW ID: ______________________________

Instructions:

- This is a **simulated practice midterm**. You will **not** be graded on your performance on this exam.
- Nevertheless, please treat this as if it is a real exam. That means that you may not discuss with your neighbors, reference outside material, or use your devices during the next 50 minute period.
- If you get stuck on a problem, consider moving on and coming back later. In the actual exam, there will likely be opportunity for partial credit.
- There are 5 problems on this exam, totaling 90 points.

1. Predicate Translation [15 points]

Let the domain of discourse be novels, comic books, movies, and TV shows. Translate the following statements to predicate logic, using the following predicates:

Novel(x) := x is a novel
Comic(x) := x is a comic book
Movie(x) := x is a movie
Show(x) := x is a TV show
Adaptation(x, y) := x is an adaptation of y

(a) (5 points) A novel cannot be adapted into both a movie and a TV show.

Solution:

$\forall x(\text{Novel}(x) \to \forall m \forall s((\text{Movie}(m) \land \text{Show}(s)) \to \neg(\text{Adaptation}(m, x) \land \text{Adaptation}(s, x)))$

(b) (5 points) Every movie is an adaptation of a novel or a comic book.

Solution:

$\forall m(\text{Movie}(m) \to \exists x(\text{Adaptation}(m, x) \land (\text{Novel}(x) \lor \text{Comic}(x))))$

(c) (5 points) Every novel has been adapted into exactly one movie.

Solution:

$\forall x(\text{Novel}(x) \to \exists m(\text{Movie}(m) \land \text{Adaptation}(m, x) \land \forall n((\text{Movie}(n) \land (n \neq m)) \to \neg\text{Adaptation}(n, x))))$

OR

$\forall x(\text{Novel}(x) \to \exists m(\text{Movie}(m) \land \text{Adaptation}(m, x) \land \forall n(\text{Adaptation}(n, x) \to (\neg\text{Movie}(n) \lor n = m))))$

OR

$\forall x(\text{Novel}(x) \to \exists m(\text{Movie}(m) \land \text{Adaptation}(m, x) \land \forall n((\text{Adaptation}(n, x) \land \text{Movie}(n)) \to (n = m))))$

*Note that a great exercise is to show that the above 3 solutions are all logically equivalent :)

2. Circuits [15 points]

The boolean function f takes in three inputs x_1, x_2, x_3 (where each is a 0 or 1 value), and outputs 1 if $(x_1 * x_2) + x_3$ is even, and 0 otherwise.

(a) (5 points) Draw a truth table for f.

Solution:

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	1

(b) (5 points) Write f as a sum-of-products expression.

Solution:

$x_1x_2x_3 + x_1x_2'x_3' + x_1'x_2x_3' + x_1'x_2'x_3'$

(c) (5 points) Write f as a products-of-sums expression.

Solution:

$(x_1' + x_2' + x_3)(x_1' + x_2 + x_3')(x_1 + x_2' + x_3')(x_1 + x_2 + x_3')$

3. Number Theory Proof [20 points]
Recall this definition of odd: $\text{Odd}(x) := \exists y(x = 2y + 1)$. Write an English proof to show that for all odd integers k, the statement $8 \mid k^2 - 1$ holds.

Hint: At some point in your proof, you'll need to show that for any integer a, $a(a + 1)$ is even. When you reach this point, feel free to break your proof up into the case where a is even, and the case where a is odd.

Solution:
Let k be an arbitrary odd integer. Then $k = 2a + 1$ for some integer a. Then $k^2 - 1 = (2a + 1)^2 - 1 = 4a^2 + 4a + 1 - 1 = 4a^2 + 4a = 4a(a + 1)$.

Consider the case where a is odd. Then $a = 2b + 1$ for some integer b. Then $k^2 - 1 = 4a(a + 1) = 4(2b + 1)(2b + 2) = 8(2b + 1)(b + 1)$. By closure of integers under multiplication and addition, $k^2 - 1 = 8c$ for an integer c. Thus in this case, $8 \mid k^2 - 1$.

Consider the case where a is even. Then $a = 2b$ for some integer b. Then $k^2 - 1 = 4a(a+1) = 4(2b)(2b+1) = 8b(2b + 1)$. By closure of integers under multiplication and addition, $k^2 - 1 = 8c$ for an integer c. Thus in this case, $8 \mid k^2 - 1$.

So in all cases, $8 \mid k^2 - 1$. Since k was an arbitrary odd integer, we have proved the claim.

4. Set Proof [20 points]

Suppose that for sets A, B, C, the facts $A \subseteq B$ and $B \subseteq C$ are given. Write an English proof to show that $B \times A \subseteq C \times C$.

Solution:

Suppose that for sets A, B, C, we have $A \subseteq B$ and $B \subseteq C$ (these are our givens). Let $x \in B \times A$ be arbitrary. Then by definition of Cartesian Product, $x = (y, z)$ for $y \in B$ and $z \in A$. Then since $y \in B$ and $B \subseteq C$, $y \in C$. Similarly since $z \in A$ and $A \subseteq B$, $z \in B$. Then since $z \in B$ and $B \subseteq C$, we have $z \in C$. Therefore we have shown that $y \in C$ and $z \in C$. Then by definition of Cartesian Product, $x \in C \times C$. Since x was arbitrary, we have shown $B \times A \subseteq C \times C$.

5. Induction [20 points]
Prove by induction that $3^n - 1$ is divisible by 2 for any integer $n \geq 1$.

Solution:

1. Let P(n) be the statement "$3^n - 1$ is divisible by 2". We prove P(n) for all integers $n \geq 1$ by induction.

2. Base Case: When $n = 1$, $3^n - 1 = 3^1 - 1 = 3 - 1 = 2$. Since $2 \mid 2$, the base case holds.

3. Inductive Hypothesis: Suppose that P(k) holds for some arbitrary integer $k \geq 1$. Then $2 \mid 3^k - 1$. Then by definition of divides, there exists some integer a such that $3^k - 1 = 2a$.

4. Inductive Step: Observe that...

$$\begin{aligned}
3^{k+1} - 1 &= 3(3^k) - 1 && \text{Definition of Exponent} \\
&= 3(3^k - 1 + 1) - 1 && \text{Subtract and Add by 1} \\
&= 3(2a + 1) - 1 && \text{By IH} \\
&= 6a + 3 - 1 && \text{Algebra} \\
&= 6a + 2 && \text{Algebra} \\
&= 2(3a + 1) && \text{Algebra}
\end{aligned}$$

Thus by definition of divides, $2 \mid 3^{k+1} - 1$. So P($k + 1$) holds.

5. Thus we have proven P(n) for all integers $n \geq 1$ by induction.