

Conceptual Review

(a) Set Definitions

Set Equality: \( A = B := \forall x (x \in A \leftrightarrow x \in B) \)
Subset: \( A \subseteq B := \forall x (x \in A \rightarrow x \in B) \)
Union: \( A \cup B := \{ x : x \in A \vee x \in B \} \)
Intersection: \( A \cap B := \{ x : x \in A \land x \in B \} \)
Set Difference: \( A \setminus B := \{ x : x \in A \land x \notin B \} \)
Set Complement: \( \overline{A} = A^C := \{ x : x \notin A \} \)
Power set: \( P(A) := \{ B : B \subseteq A \} \)
Cartesian Product: \( A \times B := \{ (a, b) : a \in A, b \in B \} \)

(b) How do we prove that for sets \( A \) and \( B \), \( A \subseteq B \)?

Solution:
Let \( x \in A \) be arbitrary... thus \( x \in B \). Since \( x \) was arbitrary, \( A \subseteq B \).

(c) How do we prove that for sets \( A \) and \( B \), \( A = B \)?

Solution:
Method 1: Use two subset proofs to show that \( A \subseteq B \) and \( B \subseteq A \).
Method 2: Use a chain of logical equivalences.

(d) What does \( \{ x \in \mathbb{Z} : x > 0 \} \) mean? Note: this notation is called "set-builder" notation.

Solution:
The set of all positive integers.

1. Examples
(a) Prove that \( A \cap B \subseteq A \cup B \).

Solution:
Let \( x \in A \cap B \) be arbitrary. Then by definition of intersection, \( x \in A \) and \( x \in B \). So certainly \( x \in A \) or \( x \in B \). Then by definition of union, \( x \in A \cup B \).

(b) Prove that \( A \cap (A \cup B) = A \cup (A \cap B) \) with a chain of equivalences proof.

Solution:
Let \( x \) be arbitrary. Observe that:

\[
x \in A \cap (A \cup B) \equiv (x \in A) \land (x \in A \cup B) \equiv (x \in A) \land ((x \in A) \lor (x \in B))
\]

Def of Intersection

Def of Union
\[(x \in A) \land (x \in B) \lor ((x \in A) \land (x \in B)) \quad \text{Distributivity} \]
\[\equiv (x \in A) \lor (x \in A) \land (x \in B) \quad \text{Idempotency} \]
\[\equiv (x \in A) \lor (x \in A \cap B) \quad \text{Def of Intersection} \]
\[\equiv x \in A \cup (A \cap B) \quad \text{Def of Union} \]

Since \(x\) was arbitrary, we have shown \(A \cap (A \cup B) = A \cup (A \cap B)\).

## 2. Set Operations

Let \(A = \{1, 2, 5, 6, 8\}\) and \(B = \{2, 3, 5\}\).

(a) What is the set \(A \cap (B \cup \{2, 8\})\)?

**Solution:**
\[\{2, 5, 8\}\]

(b) What is the set \(\{10\} \cup (A \setminus B)\)?

**Solution:**
\[\{1, 6, 8, 10\}\]

(c) What is the set \(P(B)\)?

**Solution:**
\[\{\{2, 3, 5\}, \{2, 3\}, \{2, 5\}, \{3, 5\}, \{2\}, \{3\}, \{5\}, \emptyset\}\]

(d) How many elements are in the set \(A \times B\)? List 3 of the elements.

**Solution:**
15 elements, for example \((1, 2), (1, 3), (1, 5)\).

## 3. Set Equality Proof

(a) Write an English proof to show that \(A \cap (A \cup B) \subseteq A\) for any sets \(A, B\).

**Solution:**
Let \(x\) be an arbitrary member of \(A \cap (A \cup B)\). Then by definition of intersection, \(x \in A\) and \(x \in A \cup B\). So certainly, \(x \in A\). Since \(x\) was arbitrary, \(A \cap (A \cup B) \subseteq A\).

(b) Write an English proof to show that \(A \subseteq A \cap (A \cup B)\) for any sets \(A, B\).

**Solution:**
Let \(y \in A\) be arbitrary. So certainly \(y \in A\) or \(y \in B\). Then by definition of union, \(y \in A \cup B\). Since \(y \in A\) and \(y \in A \cup B\), by definition of intersection, \(y \in A \cap (A \cup B)\). Since \(y\) was arbitrary, \(A \subseteq A \cap (A \cup B)\).

(c) Combine part (a) and (b) to conclude that \(A \cap (A \cup B) = A\) for any sets \(A, B\).
Solution:
Since \( A \cap (A \cup B) \subseteq A \) and \( A \subseteq A \cap (A \cup B) \), we can deduce that \( A \cap (A \cup B) = A \).

(d) Prove \( A \cap (A \cup B) = A \) again, but using a chain of equivalences proof instead.

Solution:
Let \( x \) be arbitrary. Observe:

\[
\begin{align*}
x \in A \cap (A \cup B) & \equiv (x \in A) \land (x \in A \cup B) & \text{Def of Intersection} \\
& \equiv (x \in A) \land ((x \in A) \lor (x \in B)) & \text{Def of Union} \\
& \equiv x \in A & \text{Absorption}
\end{align*}
\]

Since \( x \) was arbitrary, we have shown \( A \cap (A \cup B) = A \).

4. Subsets
Prove or disprove: for any sets \( A, B, \) and \( C \), if \( A \subseteq B \) and \( B \subseteq C \), then \( A \subseteq C \).
Solution:
Let \( A, B, C \) be sets, and suppose \( A \subseteq B \) and \( B \subseteq C \). Let \( x \) be an arbitrary element of \( A \). Then, by definition of subset, \( x \in B \), and by definition of subset again, \( x \in C \). Since \( x \) was an arbitrary element of \( A \), we see that all elements of \( A \) are in \( C \), so by definition of subset, \( A \subseteq C \). So, for any sets \( A, B, C \), if \( A \subseteq B \) and \( B \subseteq C \), then \( A \subseteq C \).

5. \( \cup \rightarrow \cap \)?
Prove or disprove: for all sets \( A \) and \( B \), \( A \cup B \subseteq A \cap B \).
Solution:
We wish to disprove this claim via a counterexample. Choose \( A = \{1\}, B = \emptyset \). Note that \( A \cup B = \{1\} \cup \emptyset = \{1\} \) by definition of set union. Note that \( A \cap B = \{1\} \cap \emptyset = \emptyset \) by definition of set intersection. \( \{1\} \not\subseteq \emptyset \), so the claim does not hold for these sets. Since we found a counterexample to the claim, we have shown that it is not the case that \( A \cup B \not\subseteq A \cap B \) for all sets \( A \) and \( B \).

6. Cartesian Product Proof
Write an English proof to show that \( A \times C \subseteq (A \cup B) \times (C \cup D) \).
Solution:
Let \( x \in A \times C \) be arbitrary. Then \( x \) is of the form \( x = (y, z) \), where \( y \in A \) and \( z \in C \). Then certainly \( y \in A \) or \( y \in B \). Then by definition of union, \( y \in (A \cup B) \). Similarly, since \( z \in C \), certainly \( z \in C \) or \( z \in D \). Then by definition, \( z \in (C \cup D) \). Since \( x = (y, z) \), then \( x \in (A \cup B) \times (C \cup D) \). Since \( x \) was arbitrary, we have shown \( A \times C \subseteq (A \cup B) \times (C \cup D) \).

7. Set Equality Proof
We want to prove that \( A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C) \).
(a) First prove this with a chain of logical equivalences proof.

Solution:
Let \( x \) be arbitrary. Observe:

\[
A \setminus (B \cap C) \equiv (x \in A) \land (x \not\in B \cap C) & \text{Def of Set Difference}
\]
Recall that integers are the numbers \( \ldots, -2, -1, 0, 1, 2, \ldots \), and arithmetic operations \( +, -, \div, \cdot \).

Since \( x \) was arbitrary, we have shown \( A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C) \).

(b) Now prove this with an English proof that is made of two subset proofs.

Solution:

Let \( x \in A \setminus (B \cap C) \) be arbitrary. Then by definition of set difference, \( x \in A \) and \( x \notin B \cap C \). Then by definition of intersection, \( x \notin B \) or \( x \notin C \). Thus (by distributive property of propositions) we have \( x \in A \) and \( x \notin B \), or \( x \in A \) and \( x \notin C \). Then by definition of set difference, \( x \in (A \setminus B) \) or \( x \in (A \setminus C) \). Then by definition of union, \( x \in (A \setminus B) \cup (A \setminus C) \). Since \( x \) was arbitrary, we have shown \( A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C) \).

Let \( x \in (A \setminus B) \cup (A \setminus C) \) be arbitrary. Then by definition of union, \( x \in (A \setminus B) \) or \( x \in (A \setminus C) \). Then by definition of set difference, \( x \in A \) and \( x \notin B \), or \( x \in A \) and \( x \notin C \). Then (by distributive property of propositions) \( x \in A \), and \( x \notin B \) or \( x \notin C \). Then by definition of intersection, \( x \in A \) and \( x \notin (B \cap C) \). Then by definition of set difference, \( x \in A \setminus (B \cap C) \). Since \( x \) was arbitrary, we have shown that \( (A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C) \).

Since \( A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C) \) and \( (A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C) \), we have shown \( A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C) \).

8. Constructing Sets

Use set builder notation to construct the following sets. You may use arithmetic predicates \( =, \neq, \leq, \geq \), and arithmetic operations \( +, -, \div, \cdot \).

Recall that integers are the numbers \( \ldots, -2, -1, 0, 1, 2, \ldots \), and are denote \( \mathbb{Z} \).

(a) The set of even integers.

Solution:

\[ \{2x \mid x \in \mathbb{Z}\} \text{ or } \{x \mid x = 2k, k \in \mathbb{Z}\} \text{ or } \{x \in \mathbb{Z} : 2|x\} \]

(b) The set of integers that are one more than a perfect square.

Solution:

\[ \{x^2 + 1 : x \in \mathbb{Z}\} \]

(c) The set of integers that are greater than 5.
Solution:
\{x \in \mathbb{Z} : x > 5\}

9. Making a Difference
Garrett and Shaoqi are working on their AI homework and tell you the following. Let \( G \) denote the set of AI homework questions that Garrett has not yet solved. Let \( S \) denote the set of AI homework questions that Shaoqi has not yet solved. Garrett and Shaoqi claim that \( G \setminus S = S \setminus G \).

In what circumstance is this true? In what circumstance is it false? Can you justify this (formal proof not required)?

Solution:
This is only true in the case when \( G = S \). In all other cases, \( G \setminus S \neq S \setminus G \).

Justification:
When \( G = S \), \( G \setminus S = \emptyset \) and \( S \setminus G = \emptyset \). So \( G \setminus S \neq S \setminus G \) holds.

When \( G \neq S \), then either there exists some element \( x \) such that \( x \in G \) and \( x \notin S \), or some element \( y \) such that \( y \in S \) and \( y \notin G \). Assume we are in the first case (the second case follows a similar argument). Then because \( x \in G \) and \( x \notin S \), \( x \) will be in \( G \setminus S \). However, since \( x \notin S \), \( x \) will not be in \( S \setminus G \). Thus in this case, \( G \setminus S \neq S \setminus G \).