CSE 390Z: Mathematics for Computation Workshop

Week 3 Workshop Solutions

Name: ______________________________

Collaborators: ______________________________

Conceptual Review

(a) What are the inference rules?

Solution:

Introduce $\vee$: $\frac{A}{\therefore\ A \vee B,\ B \vee A}$

Eliminate $\vee$: $\frac{A \vee B\ ;\ \neg A}{\therefore\ B}$

Introduce $\wedge$: $\frac{A\ ;\ B}{\therefore\ A \wedge B}$

Eliminate $\wedge$: $\frac{A \wedge B}{\therefore\ A\ ,\ B}$

Direct Proof: $\frac{A \Rightarrow B}{\therefore\ A \rightarrow B}$

Modus Ponens: $\frac{A\ ;\ A \rightarrow B}{\therefore\ B}$

Intro $\exists$: $\frac{P(c) \text{ for some } c}{\therefore\ \exists x P(x)}$

Eliminate $\exists$: $\frac{\exists x P(x)}{\therefore\ P(c) \text{ for a fresh } c}$

Intro $\forall$: $\frac{P(a);\ a \text{ is arbitrary}}{\therefore\ \forall x P(x)}$

Eliminate $\forall$: $\frac{\forall x P(x)}{\therefore\ P(a);\ a \text{ is arbitrary}}$

(b) Given $A \wedge B$, prove $A \vee B$

Solution:

1. $A \wedge B$ (Given)
2. A (Elim $\wedge$: 1.)
3. $A \vee B$ (Intro $\vee$: 2.)

Given $P \rightarrow R$, $R \rightarrow S$, prove $P \rightarrow S$.

1. $P \rightarrow R$ (Given)
2. $R \rightarrow S$ (Given)
   - 3.1 P (Assumption)
   - 3.2 R (Modus Ponens: 3.1, 1)
   - 3.3 S (Modus Ponens: 3.2, 2)
3. $P \rightarrow S$ (Direct Proof Rule; 3.1-3.3)

(c) How do you prove a "for all" statement? E.g. prove $\forall x P(x)$

Solution:

Show that for any **arbitrary** a in the domain, $P(a)$ holds.

(d) How do you prove a "there exists" statement? E.g. prove $\exists x P(x)$

Solution:

Show that for some specific a in the domain, $P(a)$ holds.

1. Tricky Translations

Translate the following English sentences to predicate logic. The domain is integers, and you may use $=$, $\neq$, and $>$ as predicates. Assume the predicates Prime, Composite, and Even have been defined appropriately.

(a) 2 is prime.

Solution:
Prime(2)

(b) Every positive integer is prime or composite, but not both.

Solution:
$\forall x\ ((x > 0) \rightarrow (\text{Prime}(x) \oplus \text{Composite}(x)))$
OR
$\forall x\ ((x > 0) \rightarrow [(\text{Prime}(x) \land \neg\text{Composite}(x)) \lor (\neg\text{Prime}(x) \land \text{Composite}(x))])$

(c) There is exactly one even prime.

Solution:
$\exists x((\text{Even}(x) \land \text{Prime}(x) \land \forall y[(\text{Even}(y) \land \text{Prime}(y)) \rightarrow (y = x)])$
OR
$\exists x((\text{Even}(x) \land \text{Prime}(x) \land \forall y[(y \neq x) \rightarrow \neg(\text{Even}(y) \land \text{Prime}(y))])$

(d) 2 is the only even prime.

Solution:
$\forall x\ ((x = 2) \leftrightarrow \text{Prime}(x) \land \text{Even}(x))$

2. Formal Proofs: Modus Ponens

(a) Prove that given $p \rightarrow q$, $\neg s \rightarrow \neg q$, and p, we can conclude s.

Solution:

1. $p \rightarrow q$ (Given)
2. $\neg s \rightarrow \neg q$ (Given)
3. p (Given)
4. q (Modus Ponens; 1,3)
5. $q \rightarrow s$ (Contrapositive; 2)
6. s (Modus Ponens; 5,4)

(b) Prove that given $\neg(p \lor q) \rightarrow s$, $\neg p$, and $\neg s$, we can conclude q.

Solution:

1. $\neg(p \vee q) \rightarrow s$ (Given)
2. $\neg p$ (Given)
3. $\neg s$ (Given)
4. $\neg s \rightarrow \neg\neg(p \vee q)$ (Contrapositive; 1)
5. $\neg s \rightarrow (p \vee q)$ (Double Negation; 4)
6. $p \vee q$ (Modus Ponens; 3,5)
7. q (Elim $\vee$; 6,2)

3. Formal Proofs: Direct Proof Rule

(a) Prove that given $p \rightarrow q$, we can conclude $(p \wedge r) \rightarrow q$

Solution:

1. $p \rightarrow q$ (Given)
   - 2.1 $p \wedge r$ (Assumption)
   - 2.2 p (Elim $\wedge$; 2.1)
   - 2.3 q (Modus Ponens; 2.2, 1.)
2. $(p \wedge r) \rightarrow q$ (Direct proof rule; 2.1-2.3)

(b) Prove that given $p \vee q$, $q \rightarrow r$, and $r \rightarrow s$, we can conclude $\neg p \rightarrow s$.

Solution:

1. $p \vee q$ (Given)
2. $q \rightarrow r$ (Given)
3. $r \rightarrow s$ (Given)
   - 4.1 $\neg p$ (Assumption)
   - 4.2 q (Elim $\vee$; 1, 4.1)
   - 4.3 r (Modus Ponens; 4.2, 2)
   - 4.4 s (Modus Ponens; 4.3, 3)
4. $\neg p \rightarrow s$ (Direct proof rule; 4.1-4.4)

4. Predicate Logic Formal Proof

(a) Prove that $\forall x P(x) \rightarrow \exists x P(x)$. You may assume that the domain is nonempty.

Solution:

   - 1.1. $\forall x P(x)$ (Assumption)
   - 1.2. $P(a)$ (Elim $\forall$: 1.1)
   - 1.3. $\exists x P(x)$ (Intro $\exists$: 1.2)
1. $\forall x P(x) \rightarrow \exists x P(x)$ (Direct Proof Rule, from 1.1-1.3)

(b) Given $\forall x(T(x) \rightarrow M(x))$ and $\exists x(T(x))$, prove that $\exists x(M(x))$.

Solution:

1. $\forall x(T(x) \rightarrow M(x))$ (Given)
2. $\exists x(T(x))$ (Given)
   Let r be the object that satisfies $T(r)$
3. $T(r)$ ($\exists$ elimination, from 2)
4. $T(r) \rightarrow M(r)$ ($\forall$ elimination, from 1)
5. $M(r)$ (Modus ponens, from 3 and 4)
6. $\exists x(M(x))$ ($\exists$ introduction, from 5)

(c) Given $\forall x(P(x) \rightarrow Q(x))$, prove that $(\exists x P(x)) \rightarrow (\exists y Q(y))$.

Solution:

1. $\forall x(P(x) \rightarrow Q(x))$ (Given)
   2.1. $\exists x(P(x))$ (Assumption)
   Let r be the object that satisfies $P(r)$
   2.2. $P(r)$ ($\exists$ elimination, from 2.1)
   2.3. $P(r) \rightarrow Q(r)$ ($\forall$ elimination, from 1)
   2.4. $Q(r)$ (Modus Ponens, from 2.2 and 2.3)
   2.5. $\exists y(Q(y))$ ($\exists$ introduction, from 2.4)
2. $(\exists x P(x)) \rightarrow (\exists y Q(y))$ (Direct Proof Rule, from 2.1-2.5)

5. Predicate Logic Formal Proof: Oddly Even

Write a formal proof to show: If n, m are odd, then $n + m$ is even.
Let the predicates Odd(x) and Even(x) be defined as follows where the domain of discourse is integers:

$$\text{Odd}(x) := \exists y\ (x = 2y + 1)$$

$$\text{Even}(x) := \exists y\ (x = 2y)$$

Solution:

1.	Let x be an arbitrary integer.	
2.	Let y be an arbitrary integer.	
3.1.	$\text{Odd}(x) \land \text{Odd}(y)$	[Assumption]
3.2.	$\text{Odd}(x)$	[Elim $\land$: 3.1]
3.3.	$\exists k\ (x = 2k + 1)$	[Definition of Odd, 3.2]
3.4.	$x = 2k + 1$	[Elim $\exists$: 3.3]
3.5.	$\text{Odd}(y)$	[Elim $\land$: 3.1]
3.6.	$\exists k\ (y = 2k + 1)$	[Definition of Odd, 3.5]
3.7.	$y = 2j + 1$	[Elim $\exists$: 3.7]
3.8.	$x + y = 2k + 1 + 2j + 1$	[Algebra: 3.4, 3.7]
3.9.	$x + y = 2(k + j + 1)$	[Algebra: 3.8]
3.10.	$\exists r\ (x + y = 2r)$	[Intro $\exists$: 3.9]
3.11.	$\text{Even}(x + y)$	[Definition of Even, 3.10]
3.	$\text{Odd}(x) \land \text{Odd}(y) \rightarrow \text{Even}(x + y)$	[Direct Proof Rule]
4.	$\forall m(\text{Odd}(x) \land \text{Odd}(m) \rightarrow \text{Even}(x + m))$	[Intro $\forall$: 2,3]
5.	$\forall n \forall m(\text{Odd}(n) \land \text{Odd}(m) \rightarrow \text{Even}(n + m))$	[Intro $\forall$: 1,4]

6. More Formal Proofs: Predicate Logic!

Given $\forall x\ (P(x) \lor Q(x))$ and $\forall y\ (\neg Q(y) \lor R(y))$, prove $\exists x\ (P(x) \lor R(x))$. You may assume that the domain is not empty.

Solution:

1.	$\forall x\ (P(x) \lor Q(x))$	[Given]
2.	$\forall y\ (\neg Q(y) \lor R(y))$	[Given]
3.	$P(a) \lor Q(a)$	[Elim $\forall$: 1]
4.	$\neg Q(a) \lor R(a)$	[Elim $\forall$: 2]
5.	$Q(a) \rightarrow R(a)$	[Law of Implication: 4]
6.	$\neg\neg P(a) \lor Q(a)$	[Double Negation: 3]
7.	$\neg P(a) \rightarrow Q(a)$	[Law of Implication: 5]
8.1.	$\neg P(a)$	[Assumption]
8.2.	$Q(a)$	[Modus Ponens: 8.1, 7]
8.3.	$R(a)$	[Modus Ponens: 8.2, 5]
8.	$\neg P(a) \rightarrow R(a)$	[Direct Proof]
9.	$\neg\neg P(a) \lor R(a)$	[Law of Implication: 8]
10.	$P(a) \lor R(a)$	[Double Negation: 9]
11.	$\exists x\ (P(x) \lor R(x))$	[Intro $\exists$: 10]

7. More Predicate Translation!

(a) Translate the following sentence into predicate logic:

"Every element in the array A appears at least twice"

Assume the length of the array is a positive even number n, and assume the domain of discourse is valid integer indexes of A, i.e. $\{0, 1, 2, ..., n-1\}$. You should use array notation to refer to an element of the array (i.e. $A[i]$ is the i-*th* element of the array A), and may use $=$. You should not use any other predicates you create yourself.

Solution:

$$\forall i \exists j (i \neq j \land A[i] = A[j])$$

(b) Now translate the following modified sentence into predicate logic:

"Every element in the array A appears exactly twice"

Hint: You should be able to use what you wrote for part (a), with some additions.

Solution:

The difference here is that we need to add an additional restriction so that any index that is not i or j does not have the same value as A[i] and A[j].

$$\forall i \exists j (i \neq j \land A[i] = A[j] \land \forall k((k \neq i \land k \neq j) \rightarrow A[k] \neq A[i]))$$

8. More Formal Proofs: Propositional Logic!

Use a formal proof to show that for any propositions a, b, c, the following holds.

$$a \rightarrow (b \rightarrow (c \rightarrow ((a \land b) \land c)))$$

Solution:

Left as an exercise. General idea is to introduce several assumptions, followed by several intro $\land$ steps, followed by several applications of the Direct Proof Rule.