CSE 390Z: Mathematics for Computation Workshop

Week 2 Workshop Solutions

Conceptual Review

(a) What is DNF form? What is CNF form?

Solution:

DNF and CNF are two standard forms for producing a Boolean expression, given the Boolean function values. DNF is the "sum of products" form, and CNF is the "product of sums" form.

(b) What is a domain of discourse?

Solution:

The universe of values that variables in the predicate can come from.

(c) How do you restrict to a smaller domain in a "for all"? How do you restrict to a smaller domain in an "exists"?

Solution:

If we need to restrict something quantified by a "for all", we use **implication**. If we need to restrict something quantifies by an "exists", we use **and**.

For example, suppose the domain of discourse is all animals. We translate "all birds can fly" to $\forall x(\text{Bird}(x) \to \text{Fly}(x))$. We translate "there is a bird that can fly" to $\exists x(\text{Bird}(x) \land \text{Fly}(x))$.

(d) What is the difference between $\forall x \exists y(P(x,y))$ and $\exists y \forall x(P(x,y))$?

Solution:

$\forall x \exists y(P(x,y))$ means that every x has a y that satisfies $P(x,y)$. For example, "everyone has someone that they like".

$\exists y \forall x(P(x,y))$ means that there is a specific y that satisfies $P(x,y)$ for every x. For example, "there is a person that everyone likes".

The second is stronger than the first, i.e. $\exists y \forall x(P(x,y)) \to \forall x \exists y(P(x,y))$ will always hold.

(e) What are DeMorgan's Laws for Quantifiers?

Solution:

¬∀x P(x) ≡∃x ¬P(x)
¬∃x P(x) ≡∀x ¬P(x)

1. Equivalences: Propositional Logic

Prove $((p \wedge q) \to r) \equiv (p \to r) \vee (q \to r)$ via equivalences. Note that with propositional logic, you are expected to show all steps, including commutativity and associativity.

Solution:

$$\begin{aligned}
(p \wedge q) \to r &\equiv \neg(p \wedge q) \vee r && \text{Law of Implication}\\
&\equiv (\neg p \vee \neg q) \vee r && \text{De Morgan's Law}\\
&\equiv (\neg p \vee \neg q) \vee (r \vee r) && \text{Idempotency}\\
&\equiv \neg p \vee (\neg q \vee (r \vee r)) && \text{Associativity}\\
&\equiv \neg p \vee ((\neg q \vee r) \vee r) && \text{Associativity}\\
&\equiv \neg p \vee (r \vee (\neg q \vee r)) && \text{Commutativity}\\
&\equiv \neg p \vee (r \vee (q \to r)) && \text{Law of Implication}\\
&\equiv (\neg p \vee r) \vee (q \to r) && \text{Associativity}\\
&\equiv (p \to r) \vee (q \to r) && \text{Law of Implication}
\end{aligned}$$

2. Equivalences: Boolean Algebra

(a) Prove $p' + p \cdot q + q' \cdot p = 1$ via equivalences. Note that with Boolean Algebra, you may skip commutativity associativity in the steps that you show.

Solution:

$$\begin{aligned}
p' + p \cdot q + q' \cdot p &\equiv p' + p \cdot q + p \cdot q' && \text{Commutativity}\\
&\equiv p' + p \cdot (q + q') && \text{Distributivity}\\
&\equiv p' + p \cdot 1 && \text{Complementarity}\\
&\equiv p' + p && \text{Identity}\\
&\equiv p + p' && \text{Commutativity}\\
&\equiv 1 && \text{Complementarity}
\end{aligned}$$

(b) Prove $(p' + q) \cdot (q + p) = q$ via equivalences.

Solution:

$$\begin{aligned}
(p' + q) \cdot (q + p) &\equiv (p' + q) \cdot q + (p' + q) \cdot p && \text{Distributivity}\\
&\equiv q + (p' + q) \cdot p && \text{Absorption}\\
&\equiv q + qp && \text{Absorption}\\
&\equiv q && \text{Absorption}
\end{aligned}$$

3. DNFs and CNFs

Consider the following boolean functions $A(p, q, r)$ and $B(p, q, r)$.

p	q	r	$A(p, q, r)$	$B(p, q, r)$
1	1	1	0	1
1	1	0	0	1
1	0	1	1	1
1	0	0	0	0
0	1	1	1	0
0	1	0	1	1
0	0	1	0	1
0	0	0	0	0

(a) Write the DNF (sum of products) and CNF (product of sums) expressions for $A(p, q, r)$.

Solution:

DNF: $pq'r + p'qr + p'qr'$

CNF: $(p' + q' + r')(p' + q' + r)(p' + q + r)(p + q + r')(p + q + r)$

(b) Write the DNF (sum of products) and CNF (product of sums) expressions for $B(p, q, r)$.

Solution:

DNF: $pqr + pqr' + pq'r + p'qr' + p'q'r$

CNF: $(p' + q + r)(p + q' + r')(p + q + r)$

4. Domains of Discourse

For the following, find a domain of discourse where the following statement is true and another where it is false. Note that for the arithmetic symbols to make sense, the domains of discourse should be sets of numbers.

(a) $\exists x(2x = 0)$

Solution:

True domain: Any set of numbers that includes 0; e.g. all natural numbers.
False domain: Any set of numbers that doesn't include 0; e.g. all integers greater than 0.

(b) $\forall x \exists y(x + y = 0)$

Solution:

True domain: Any set of numbers that includes additive inverses; e.g. all integers.
False domain: Any set of numbers that doesn't include additive inverses; e.g. all positive integers.

(c) $\exists x \forall y(x + y = y)$

Solution:

True domain: Any set of numbers that includes 0; e.g. all natural numbers (if $x = 0$, the statement holds for all y).
False domain: Any set of numbers that doesn't include 0; e.g. all integers greater than 0.

5. Predicate Logic Gotchas

Let the domain of discourse be all animals. Let Cat(x) ::= "x is a cat" and Blue(x) ::= "x is blue". Translate the following statements to English.

(a) $\forall x(\text{Cat}(x) \wedge \text{Blue}(x))$

Solution:

All animals are blue cats.

(b) $\forall x(\text{Cat}(x) \rightarrow \text{Blue}(x))$

Solution:

All cats are blue.

(c) $\exists x(\text{Cat}(x) \wedge \text{Blue}(x))$

Solution:

There exists a blue cat.

Kabir translated the sentence "there exists a blue cat" to $\exists x(\text{Cat}(x) \rightarrow \text{Blue}(x))$. This is wrong! Let's understand why.

(d) Use the Law of Implications to rewrite Kabir's translation without the $\rightarrow$.

Solution:

$\exists x(\neg\text{Cat}(x) \vee \text{Blue}(x))$

(e) Translate the predicate from (d) back to English. How does this differ from the intended meaning?

Solution:

Translation: There exists an animal that is not a cat, or is blue.

The difference: If there was even one non-cat animal in the universe (e.g. a single dog), this condition would be satisfied. Similarly, if there was even one blue animal in the universe, this condition would be satisfied. So, this is a very different condition than "there exists a blue cat".

(f) This is a warning to be very careful when including an implication nested under an exists! It should almost always be avoided, unless there is a forall involved as well. (Nothing to write for this part).

6. English to Predicate Logic

Express the following sentences in predicate logic. The domain of discourse is penguins. You may use the following predicates: Love(x, y) ::= "x loves y", Dances(x) ::= "x dances", Sings(x) ::= "x sings".

(a) There is a penguin that every penguin loves.

Solution:

$\exists x\forall y(\text{Loves}(y, x))$

(b) All penguins that sing love a penguin that does not sing.

Solution:

$\forall x(\text{Sings}(x) \rightarrow \exists y(\neg\text{Sings}(y) \wedge \text{Loves}(x, y)))$

(c) There is exactly one penguin that dances.

Solution:

$\exists x(\text{Dances}(x) \wedge \forall y((y \neq x) \rightarrow \neg\text{Dances}(y)))$

(d) There exists a penguin that loves itself, but hates (does not love) every other penguin.

Solution:

$\exists x(\text{Loves}(x, x) \wedge \forall y((y \neq x) \rightarrow \neg\text{Loves}(x, y)))$

7. Predicate Logic to English

Translate the following sentences to English. Assume the same predicates and domain of discourse as the previous problem.

(a) $\neg\exists x(\text{Dances}(x))$

Solution:

No penguins dance.

(b) $\exists x\forall y(\text{Loves}(x, y))$

Solution:

There is a penguin that loves all penguins.

(c) $\forall x(\text{Dances}(x) \rightarrow \exists y(\text{Loves}(y, x)))$

Solution:

All penguins that dance have a penguin that loves them.

(d) $\exists x\forall y((\text{Dances}(y) \wedge \text{Sings}(y)) \rightarrow \text{Loves}(x, y))$

Solution:

There exists a penguin that loves all penguins who dance and sing.

8. Predicate Challenge!!

Translate "You can fool all of the people some of the time, and you can fool some of the people all of the time, but you can't fool all of the people all of the time" into predicate logic. Then, negate your translation. Then, translate the negation back into English.

Hint: Let the domain of discourse be all people and all times, and let $P(x, y)$ be the statement "You can fool person x at time y". You can get away with not defining any other predicates if you use P.

Solution:

The original statement can thus be translated as

$$(\forall x \exists y P(x, y)) \wedge (\exists z \forall a P(z, a)) \wedge (\neg \forall b \forall c P(b, c))$$

The negation of this statement, in predicate logic, is

$$(\exists x \forall y \neg P(x, y)) \vee (\forall z \exists a \neg P(z, a)) \vee (\forall b \forall c P(b, c))$$

which in English translates to

"There are some people you can't ever fool, or all people have some time at which you can't fool them, or you can fool everyone at all times"