390Z Week 8: Strong & Structural Induction

Strong Induction

For any predicate $P(n)$,

If $P(0)$ is true and $\underbrace{(P(0) \wedge P(1) \wedge P(2) \wedge \ldots \wedge P(n))}_{\text{This is the only difference w/ normal induction.}} \rightarrow P(n+1)$

then $P(0), P(1), P(2), P(3), P(4), \ldots$ (on forever)

So $P(n)$ holds for any $n \geq 0$.

This is just domain restriction. It is the same as

Rule:
$$\frac{P(0) \quad \forall k(\forall j(0 \leq j \leq k \rightarrow P(j)) \rightarrow P(k+1))}{\therefore \forall n\, P(n)}$$

What changes in our proofs?

(Key Differences in Red) (Non-boiler plate parts in Blue)

1. Define $P(n) :=$ 'blah'. We will show $P(n)$ holds for all integers $n \geq b$ by strong induction.
2. Base Case(s) – Often strong induction proofs will require multiple base cases.
3. Inductive Hypothesis: Assume that $P(j)$ holds for every integer j where $b \leq j \leq k$ for some arbitrary integer $k \geq b$.
4. Inductive Step: Show that the (now stronger) IH implies that $P(k+1)$ is true. Make sure to cite the IH!
5. Conclusion: Then by the Principle of Induction, $P(n)$ is true for all integers $n \geq b$ as claimed.

Ex.

I can order chicken nuggets in groups of 4 or 5. (Problem modified from Robbie Weber's sp22 311 Lecture 18)

Theorem I can order exactly n chicken nuggets for any $n \geq 12$.

Spoof.

Define $P(n) :=$ 'I can order exactly n chicken nuggets in just groups of 4 and 5'

We prove $P(n)$ holds for all integers $n \geq 12$ by induction on n.

Base Case: $12 = 3 \cdot 4$, so we can make 12 using 3 orders of 4. So $P(12)$ holds.

Inductive Hypothesis: Suppose $P(k)$ for some integer $k \geq 12$.

Inductive Step: By IH, we can order k chicken nuggets in orders of 4 and 5. Replace one order of 4 with an order of 5. Then we have $P(k+1)$. This is wrong! Why?

When you're stuck on an induction proof, make your inductive hypothesis stronger. One way to do this is strong induction.

Remark: (bonus info) Strong induction is not a seperate axiom from weak induction. In fact, you can rephrase any strong induction proof as weak induction by redefining $P(n)$ to be '(the claim) holds for any integer $n \geq b$'. Which you may notice is effectively the same modification we made to the inductive step. Strong Induction is kind of just a convenient abbreviation for this trick. Do not try to write your strong induction proofs this way – it gets very messy and confuses your reader. But it's good to know that strong induction logically follows from normal induction.

What was wrong with the Spoof?

P(k) tells us we can order k chicken nuggets, but it doesn't tell us how.

What if we have P(15)? We can get 15 nuggets with 3 orders of 5 nuggets. (there is no way using a group of 4!) Notice 15−4=11, which is not divisible by 5 or 4.

But then we don't have an order of 4 we can change to an order of 5.

What are our options?

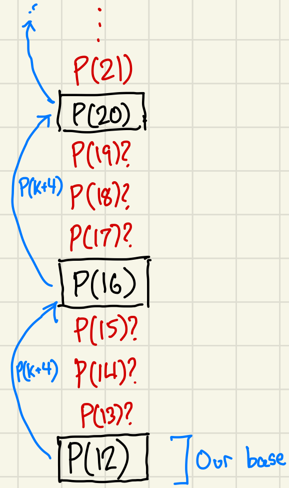
Whenever you are stuck in an induction proof, always see if you can strengthen the IH somehow.

You can sometimes avoid this by making a really complicated Inductive Step, but that's very messy.

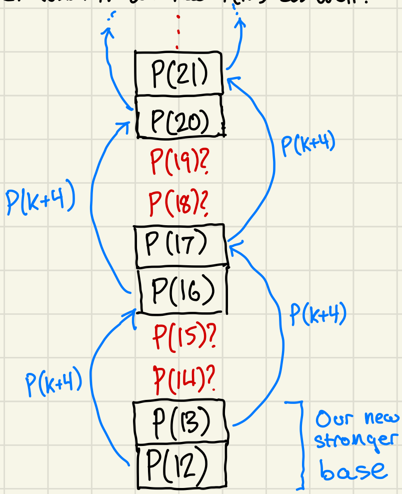
Observation: If we can order k chicken nuggets, we can order k+4 chicken nuggets by just adding an order of 4.

Strong Induction Visualization (building a tower)

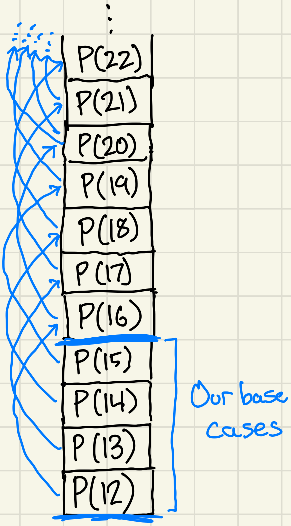
We have an easy way to order k+4 chicken nuggets. If we apply this over and over, we can "build a tower".



Notice we have a gap of 3 between the blocks of our tower. How can we fill them? By making a stronger base. What if we had P(13) as well?



Then we've filled part of the gap in the tower. If we can fill in our base cases to cover the first gap of the tower, then our inductive step, $P(k) \to P(k+4)$ fills in all the gaps in the tower for us. An equivalent way of thinking about it is $P(k-3) \to P(k+1)$ (this is the form we will use for our proof).



With a full base, our inductive steps now can build a complete tower! We've removed the gaps. Determining the size of the gap is a strategy for finding how many base cases you need.

Pf. (strong induction)

Define $P(n)$:= 'I can order n chicken nuggets in just groups of 4 and 5'.

We prove $P(n)$ holds for all integers $n \geq 12$ using strong induction on n.

Base Cases: (Here we are building the base of the tower)

- $n=12$: 3 orders of 4, $3 \cdot 4 = 12$. So $P(12)$ holds.
- $n=13$: 2 orders of 4, 1 order of 5. $2 \cdot 4 + 5 = 13$. So $P(13)$ holds.
- $n=14$: 1 order of 4, 2 orders of 5. $1 \cdot 4 + 2 \cdot 5 = 14$. So $P(14)$ holds.
- $n=15$: 3 orders of 5, $3 \cdot 5 = 15$. So $P(15)$ holds.

the highest base case

Inductive Hypothesis: Suppose $P(j)$ holds for all integers $j \leq k$ for some integer $k \geq 15$.

Can also say 'Suppose $P(12) \wedge P(13) \wedge \ldots \wedge P(k)$ for some integer $k \geq 15$'.

Inductive Step:

Notice we require $k \geq 15$, and $15-3=12$, which we have a base case for. If we required $k \geq 12$, then this would be a bug since $12-3=9$ isn't one of our base cases.

By the IH, $P(k-3)$ must hold. Then if we can order $k-3$ chicken nuggets, we can order one more group of 4 to get $k+1$ nuggets.

Thus, the IH $\rightarrow P(k+1)$.

Then we have shown $P(n)$ for all $n \geq 12$ by the principle of induction.

$\square$

Recursively Defined Structures

Even numbers:

base: $2 \in S$

recursive: $(x \in S) \to (x+2 \in S)$

ex. $2 \in S \to 2+2=4 \in S$

$4 \in S \to 4+2=6 \in S$

$6 \in S \to 6+2=8 \in S$

$\vdots$

on forever

Binary Trees:

Recall, binary trees are structures of nodes where each node has at most 2 children.

A node is a leaf if it has no children.

A binary tree is perfect if every non-leaf has exactly 2 children.

base: $\bullet$ (childless node) $\in S$

recursive:

Step 1. If $\bullet \in S$ and $\bullet \in S$, then  (a node w/ children $\bullet$ and $\bullet$) $\in S$.

Step 2. If $\bullet \in S$, then  $\in S$.

Step 3. If $\bullet \in S$, then  $\in S$.

ex.

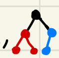
$\bullet \in S$ and $\bullet \in S$ (both are the base node, we can reuse elements)

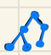
By step 1,  $\in S$

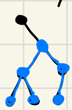
$\bullet \in S$, so by step 2,  $\in S$

$\bullet \in S$, so by step 3,  $\in S$

 $\in S$ and  $\in S$

By Step 1,  $\in S$

 $\in S$, so by step 3,

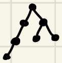
 $\in S$

Definition: The height of a binary tree is the number of edges between the root and the lowest node.

ex.

• has height 0.

 and  have height 1.

 has height 3.

Claim: A binary tree of height h has at most $2^{h+1}-1$ nodes.

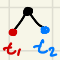
Pf. Define $P(t)$:= 'If tree t has height h, then t has at most $2^{h+1}-1$ nodes.'

We prove $P(t)$ holds for any binary tree t by structural induction.

Base Case: • has height 0 and 1 node. $2^{0+1}-1 = 2-1 = 1$, so $P(\cdot)$ holds.

Inductive Hypothesis: Suppose $P(t_1)$ and $P(t_2)$ for some binary trees t_1, t_2.

Inductive Step:

Step 1. Let $t =$ , the tree constructed by recursive step 1.

Let $h = \text{height}(t)$.

Let $n = \text{nodes}(t)$.

h is decided by the taller branch.

$h = \max\{\text{height}(t_1), \text{height}(t_2)\} + 1$

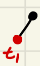
$n = \text{nodes}(t_1) + \text{nodes}(t_2)$

Goal: $n \le 2^{h+1} - 1$

Note: abbreviate $\text{height}(t_1) = h_1$, $\text{height}(t_2) = t_2$

$$
\begin{aligned}
n &= \text{nodes}(t_1) + \text{nodes}(t_2) + 1 \\
&\le 2^{\text{height}(t_1)+1} - 1 + 2^{\text{height}(t_2)+1} - 1 + 1 \quad \text{By IH } (P(t_1) \wedge P(t_2)) \\
&\le 2^{\max(h_1+1, h_2+1)} - 1 + 2^{\max(h_1+1, h_2+1)} \\
&\le 2(2^h) - 1 \\
&\le 2^{h+1} - 1
\end{aligned}
$$

So, $P(t)$ holds for step 1.

Step 2.  has height of $1 + \text{height}(t_1)$, and has $\text{nodes}(t_1) + 1$ # of nodes.

Step 3 is identical.

By the IH, $\text{nodes}(t_1) \le 2^{\text{height}(t_1)+1} - 1$.

$$
\begin{aligned}
\text{Then, nodes}(t_1) &\le (2^{\text{height}(t_1)+1} - 1) + 1 \\
&\le 2^{\text{height}(t_1)+1} \qquad \text{notice height}(t_1) = 1 + \text{height}(t_1) \\
&\le 2^{\text{height}(t_1)} \\
&\le 2^h \\
&\le 2^h \cdot 2 - 1 \qquad \text{since } 2^h \ge 1\text{, so } 2 \cdot 2^h \text{ increase } 2^h \text{ by at least } 1. \\
&\le 2^{h+1} - 1
\end{aligned}
$$

So, $P(t_1)$ holds for Step 2.

So, $P(t)$ holds for all binary trees t by structural induction. $\square$