CSE390D—Introduction to Discrete Math
Key to Final

1. (12 points) The table below contains English descriptions in terms of x and y of relations on the set Z^+ (x being related to y iff the described condition holds). Fill in the table indicating whether each relation is reflexive, symmetric, antisymmetric or transitive.

Relation	**reflexive?**	**symmetric?**	**antisymmetric?**	**transitive?**
$x - y = 10$	no	no	yes	no
$x - y < 10$	yes	no	no	no
$x^y = y^x$	yes	yes	no	yes

2. (6 points)

a. (3 points) Suppose that in a certain game a player known as "the attacker" rolls one 6-sided die and the opponent, known as "the defender," rolls one 6-sided die. The numbers rolled are compared and whichever player has the higher number wins. If there is a tie, the defender wins. If the players use fair dice, what is the probability that the defender wins in this scenario?

Solution: There are 6 ties where the defender wins and 30 cases without a tie where the defender wins half of the cases (15 cases) out of the 36 cases total.

$$\frac{21}{36}$$

b. (3 points) Consider a variation on the scenario above in which the attacker rolls two dice and the defender rolls one die. The larger of the two numbers rolled by the attacker is compared against the number rolled by the defender. Whichever player has the higher number wins and the defender wins if there is a tie. If the players use fair dice, what is the probability that the defender wins in this scenario?

Solution: When the defender rolls a 6, he wins against all 36 of the other player's rolls. When he rolls a 5, he wins against 25. When he rolls a 4, he wins against 16. And so on. So he wins in (36 + 25 + 16 + 9 + 4 + 1) of the 6^3 (216) cases.

$$\frac{91}{216}$$

3. (10 points) In trying to prove that marijuana is a "gateway drug," organizations like the Center on Addiction and Substance Abuse have often used bogus statistics that make no sense mathematically to argue that a person is many times more likely to have used a drug like heroin if they use marijuana (multiples that would often put the probability above 1). Let's compute some real answers. In the US, 1.5% of the population have used heroin at some point in their lives. Among those who have used

heroin, 99.8% have also used marijuana at some point in their lives. Among those who have never used heroin, 61% have never used marijuana.

a. (5 points) What is the probability that someone has used heroin at some point in their lives given that they have used marijuana at some point in their lives?

Solution: Let E be the event that a person has used marijuana and let F be the event that a person has used heroin. We are given:

$$P(F) = 0.015, P(\overline{F}) = 0.985$$
$$P(E \mid F) = 0.998, P(\overline{E} \mid F) = 0.002$$
$$P(E \mid \overline{F}) = 0.39, P(\overline{E} \mid \overline{F}) = 0.61$$

From Baye's Rule we know that:

$$P(F \mid E) = \frac{P(E \mid F)P(F)}{P(E \mid F)P(F) + P(E \mid \overline{F})P(\overline{F})} = \frac{0.998 \cdot 0.015}{0.998 \cdot .015 + 0.39 \cdot 0.985} \cong 0.0375$$

Notice that this probability is around 2.5 times the probability of F. So there is some evidence of a gateway effect here, but the technique used by CASA greatly exaggerates it. For example, google the phrase "85 times marijuana cocaine" and see the many sources that quote a statistic that a high school student is 85 times more likely to use cocaine if they have used marijuana (a number that would exceed a probability of 1).

b. (5 points) What is the probability that someone has never used heroin given that they have never used marijuana?

Solution: We have the same values as above and again using Baye's Rule:

$$P(\overline{F} \mid \overline{E}) = \frac{P(\overline{E} \mid \overline{F})P(\overline{F})}{P(\overline{E} \mid \overline{F})P(\overline{F}) + P(\overline{E} \mid F)P(F)} = \frac{0.61 \cdot 0.985}{0.61 \cdot .985 + 0.002 \cdot 0.015} \cong 0.99995$$

This shows that there is a much stronger argument to be made that knowing that someone has never used marijuana is a really good predictor that they have never used heroin.

4. (8 points) Prove using mathematical induction that:

$$3 + \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n} = \frac{1}{3}\left(10 - \frac{1}{10^n}\right)$$

Provide a formal definition for the overall proposition being proved (P(n)) using summation notation and indicate the domain for n. Then provide a proof by induction, clearly indicating where you are applying the inductive hypothesis.

Solution: The overall proposition is:

P(n) : $\sum_{i=0}^{n} \frac{3}{10^i} = \frac{1}{3}\left(10 - \frac{1}{10^n}\right)$ for all n ∈ N

We first prove the base case for n = 0:

$$3 = \frac{1}{3} \cdot 9 = \frac{1}{3} \cdot (10 - 1) = \frac{1}{3}\left(10 - \frac{1}{10^0}\right)$$

Then we assume that P(k) holds for some k in N:

$$\sum_{i=0}^{k} \frac{3}{10^i} = \frac{1}{3}\left(10 - \frac{1}{10^k}\right)$$

and we show that P(k + 1) is true:

$$\sum_{i=0}^{k+1} \frac{3}{10^i} = \frac{1}{3}\left(10 - \frac{1}{10^{k+1}}\right)$$

We start by expanding the summation for k + 1:

$$\sum_{i=0}^{k+1} \frac{3}{10^i} = \sum_{i=0}^{k} \frac{3}{10^i} + \frac{3}{10^{k+1}}$$

By the inductive hypothesis, we know that this equals:

$$\frac{1}{3}\left(10 - \frac{1}{10^k}\right) + \frac{3}{10^{k+1}}$$
$$= \frac{1}{3}\left(10 - \frac{1}{10^k} + \frac{9}{10^{k+1}}\right)$$
$$= \frac{1}{3}\left(10 - \frac{10}{10^{k+1}} + \frac{9}{10^{k+1}}\right)$$
$$= \frac{1}{3}\left(10 - \frac{1}{10^{k+1}}\right)$$

This completes the proof, so P(n) holds for all n in N.

5. (4 points) Consider the relation R on Z^+ composed of all ordered pairs (x, y) for which:

   $|x - 2| \leq |y - 2|$

   Is R a partial ordering? Why or why not?

   Solution: It is not a partial ordering because the relation contains the pairs (1, 3) and (3, 1), which means that it is not antisymmetric. If the pair (3, 1) is removed, the relation is reflexive, antisymmetric, and transitive, which would make it a partial ordering.

6. (8 points) Prove that the difference of the cubes of two integers is always 2 more than a multiple of 6 when the difference of the integers is 2.

   Solution: We can prove this directly. Assume that the difference of two integers is 2. Let the two integers be x and x + 2. Then computing the difference of their cubes:

   $(x + 2)^3 - x^3 = x^3 + 6x^2 + 12x + 8 - x^3 = 6x^2 + 12x + 8 = 6(x^2 + 2x + 1) + 2$

   Thus, the difference of the cubes of the integers is 2 more than a multiple of 6. This completes the proof.

7. (8 points) The game of Yahtzee involves rolling five 6-sided dice and examining the numbers rolled. We define an outcome to be the sequence of numbers rolled where order matters, as in (3, 4, 3, 3, 2), which is considered “three of a kind” in Yahtzee. There are a total of 6^5 outcomes.

a. (2 points) How many outcomes are composed of five different numbers?

   Solution: This is a 5-permutation of the 6 possible rolls:
   P(6, 5)

b. (2 points) How many outcomes have three of a kind (i.e., include three of one number and one each of two other numbers)?

   Solution: We pick the three-of-a-kind number, then pick the other two numbers, then pick where to place the other two numbers:

   $$\binom{6}{1}\binom{5}{2}P(5,2)$$

c. (2 points) How many outcomes have two pair (i.e., include a pair of one number, a pair of another number, and one of a third number)?

   Solution: We pick which two numbers are pairs, pick were to put the first pair, pick where to place the second pair, and pick the extra number for the final position:

   $$\binom{6}{2}\binom{5}{2}\binom{3}{2}\cdot 4$$

d. (2 points) If we redefine an outcome so that the order of the dice doesn’t matter, just a count of how many of each number occurs, then how many different outcomes are there?

   Solution: This is a stars and bars problem with 5 stars and 5 bars:

   $$\binom{10}{5}$$

8. (10 points) Prove or disprove that the product of two consecutive integers is never 4 more than a multiple of 7.

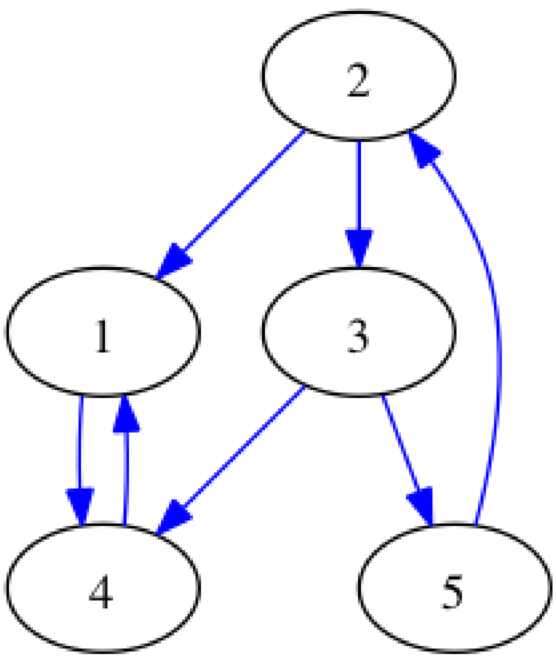
Solution: The proposition is true. Proving that the product is never 4 more than a multiple of 7 reduces to proving that the product is never congruent to 4 in modulo 7. We can prove this by cases. There are 7 cases for consecutive integers in modulo 7:

$0 \cdot 1 \equiv 0$
$1 \cdot 2 \equiv 2$
$2 \cdot 3 \equiv 6$
$3 \cdot 4 \equiv 5$
$4 \cdot 5 \equiv 6$
$5 \cdot 6 \equiv 2$
$6 \cdot 0 \equiv 0$

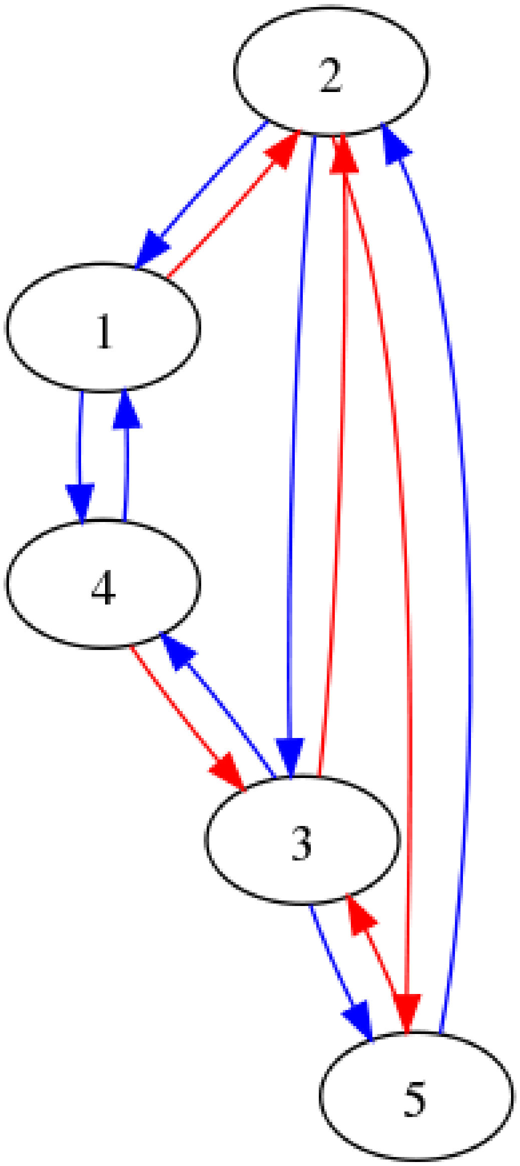
In no case do we obtain 4. This completes the proof that the product of two consecutive integers is never 4 more than a multiple of 7.

9. (8 points) Let R be the relation { (2, 1), (2, 3), (3, 5), (5, 2), (3, 4), (1, 4), (4, 1)} defined on the set {1, 2, 3, 4, 5}.

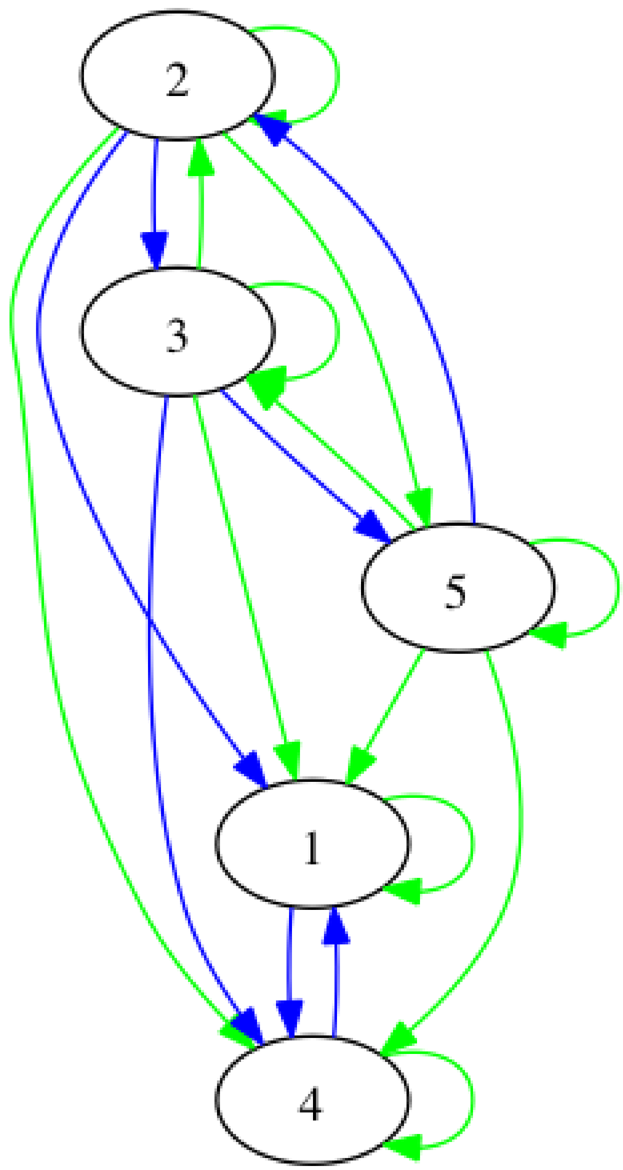
a. (2 points) Draw the graph of R.



b. (2 points) Draw the graph of the symmetric closure of R.



c. (4 points) Draw the graph of the transitive closure of R.



10. (10 points) A certain exam has 20 true/false questions and 15 multiple-choice questions. Each multiple-choice question has five possible answers (a, b, c, d, or e). An answer sheet is a form that records the answers to each of these 35 questions.

a. (2 points) How many ways are there to fill out the answer sheet if every question is answered?

Solution: For each of 20 true/false, there are 2 choices, and for each of 15 multiple choice, there are 5 choices:
$2^{20}\ 5^{15}$

b. (2 points) How many ways are there to fill out the answer sheet if every question is answered and half of the true/false questions are answered "true"?

Solution: Pick the 10 true answers and then there are 15 multiple choice with 5 choices:

$$\binom{20}{10} \cdot 5^{15}$$

c. (2 points) How many ways are there to fill out the answer sheet if every question is answered and at least two true/false questions are answered "true"?

Solution: (total) - (# with 1 true answer) - (# with 0 true answers)

$$2^{20} \cdot 5^{15} - \binom{20}{1} \cdot 5^{15} - 5^{15}$$

d. (2 points) How many ways are there to fill out the answer sheet if some or all of the questions can be left blank?

Solution: : Now there are 3 choices for each true/false and 6 for each multiple choide:

$3^{20}\ 6^{15}$

e. (2 points) Suppose that every multiple-choice question is answered for a particular answer sheet and that we compute a distribution of answers (how many "a" answers, how many "b" answers, and so on). How many different distributions are possible?

Solution: This is a stars and bars problem with 4 bars and 15 stars

$$\binom{19}{4}$$

11. (8 points) Prove or disprove that the difference of the squares of two positive integers is prime only if the two integers have a difference of 1 (e.g., $6^2 - 5^2 = 11$ and $7^2 - 6^2 = 13$).

Solution: The proposition is true and we can prove it directly. Let x and y be two positive integers. Then $x^2 - y^2 = (x - y)(x + y)$. By the Fundamental Theorem of Arithmetic, this is prime only if $(x - y)$ is 1 or if $(x + y)$ is 1 (otherwise there would be two factors of the number). If x and y are both positive, $(x + y)$ cannot be 1. Therefore, $(x - y)$ must be 1 if $(x^2 - y^2)$ is prime. This completes the proof.

12. (8 points) Two different people are each asked to pick a number between 1 and 10. Assume that each number is equally likely to be picked by each person.

a. (2 points) What is the probability that the sum of the two numbers picked is divisible by 2 or divisible by 7 (or both)?

Solution: 50 of the 100 combinations are divisible by 2, 13 are divisible by 7, 7 are divisible by both. Using inclusion/exclusion we find (50 + 13 - 7) that are divisible by one or the other.

$$\frac{56}{100}$$

b. (2 points) What is the probability that the product of the two numbers picked is the product of exactly 2 different primes? In other words, what is the probability that the prime factorization of the product can be expressed as p * q where p and q are primes with p not equal to q?

Solution: Four pairs produce a product of 6 ((2, 3), (3, 2), (1, 6), (6, 1)). Four pairs produce a product of 10 ((2, 5), (5, 2), (1, 10), (10, 1)). And there are two each with products of 14, 15, 21, and 35. Therefore, there are 16 cases total where the product is the product of two primes.

$$\frac{16}{100}$$

c. (2 points) Suppose that the procedure described in the previous paragraph is performed with 23 different pairs of people to see how often the product of the numbers they pick is exactly 2 different primes. What is the probability that this happens at most once?

Solution: We can think of this as a Bernoulli trial were a success p is getting such a pair. The probability of this happening at most once is the sum of the probability of 0 successes and 1 success.

$$\left(\frac{84}{100}\right)^{23} + 23\left(\frac{16}{100}\right)\left(\frac{84}{100}\right)^{22}$$

d. (2 points) What is the expected value of the sum of the two numbers picked?

Solution: The expected sum for one number is the sum of 1 through 10 divided by 10, which is 5.5. Thus, because of the linearity of expectation, the expected value for the sum of the two numbers is 11.

13. (bonus, 1 point) Name one of your favorite prime numbers and briefly describe why you like it.

Solution: (from Stuart) My favorite is 23, because I was born on the 23rd and 23 shows up in many interesting places (number of human chromosome pairs, number of Hilbert problems, movie title, etc).