CSE390D—Introduction to Discrete Math
Key to Sample Final

1. (14 points) The table below contains English descriptions in terms of x and y of relations on the set Z^+ (x being related to y iff the described condition holds). Fill in the table indicating whether each relation is reflexive, symmetric, antisymmetric or transitive.

Relation	reflexive?	symmetric?	antisymmetric?	transitive?
$x + y = 2$	no	yes	yes	yes
$(x - 5)^2 = (y - 5)^2$	yes	yes	no	yes
$xy = 9$	no	yes	no	no
$2x = 3y$	no	no	yes	no
$2 \mid xy$	no	yes	no	no
$2 \mid x(y + 1)$	yes	no	no	yes
x mod y = 0	yes	no	yes	yes

2. (8 points)
   a. (4 points) A certain high school has 145 women and 138 men in its graduating class. The graduating students select one man and one woman from their class for each of the following categories: most likely to succeed, most likely to be remembered, most scholarly, and most friendly. If nobody is allowed to win more than one category, how many different outcomes are possible?

      Solution: This involves choosing 4 women and choosing 4 men. Duplicates are not allowed and order matters, so the answer is:
      $P(145, 4) \cdot P(138, 4)$

   b. (4 points) You are helping to organize a banquet for 38 people. Each person has chosen one of four entrées (chicken, beef, vegetarian, and fish). Your job is to tell the catering service how many of each entrée to bring. How many possible orders might you place with the catering company?

      Solution: This is a stars and bars problem where we are choosing 38 items from a set of 4 with duplication allowed and where order doesn't matter. The number of outcomes is:
      $$\binom{41}{3}$$

3. (8 points) Prove using mathematical induction that:

$$3 + 7 + 11 + 15 + 19 + \cdots + (4n - 1) = 2n^2 + n$$

Provide a formal definition for the overall proposition being proved (P(n)) using summation notation and indicate the domain for n. Then provide a proof by induction, clearly indicating where you are applying the inductive hypothesis.

Solution: The overall proposition is:

$$\text{P(n)}: \sum_{i=1}^{n} 4i - 1 \quad = 2n^2 + n \text{ for all n} \in \text{Z}^+$$

We first prove the base case for n = 1:

$$3 = 2 + 1 = 2 \cdot 1^2 + 1$$

Then we assume that P(k) holds for some k in Z+:

$$\sum_{i=1}^{k} 4i - 1 \quad = 2k^2 + k$$

and we show that P(k + 1) is true:

$$\sum_{i=1}^{k+1} 4i - 1 \quad = 2(k+1)^2 + (k+1)$$

We start by expanding the summation for k + 1:

$$\sum_{i=1}^{k+1} 4i - 1 \quad = \left(\sum_{i=1}^{k} 4i - 1 \right) + 4(k+1) - 1$$

By the inductive hypothesis, we know that this equals:

$$\begin{aligned} & 2k^2 + k + 4(k+1) - 1 \\ & = 2k^2 + 5k + 3 \\ & = (2k^2 + 4k + 2) + k + 1 \\ & = 2(k^2 + 2k + 1) + (k+1) \\ & = 2(k+1)^2 + (k+1) \end{aligned}$$

This completes the proof, so P(n) holds for all n in Z^+.

4. (8 points) Genetic sequences are composed of the letters "A", "G", "T", and "C".

   a. (2 points) How many genetic sequences of length 10 have exactly two G's and exactly two C's?

      Solution: We can choose the positions of the G's, then the position of the C's, then fill in the rest of the sequence:
      $$\binom{10}{2}\binom{8}{2}2^6$$

   b. (4 points) How many genetic sequences of length 10 have exactly two G's or exactly two C's (or both)?

      Solution: We can use the principle of inclusion/exclusion to compute the number of sequences that have exactly two G's plus the number of sequences that have exactly two C's minus the number that satisfy both requirements:
      $$\binom{10}{2}\cdot 3^8+\binom{10}{2}\cdot 3^8-\binom{10}{2}\binom{8}{2}\cdot 2^6$$

   c. (2 points) If we count how many of each letter occur in a genetic sequence of length 10, how many different answers might we get (where an answer involves just the counts for each letter)?

      Solution: This is a stars and bars problem where we are choosing 10 items from a set of 4 where duplication is allowed and order doesn't matter, so the answer is:
      $$\binom{13}{3}$$

5. (8 points) Prove that there are no three consecutive positive integers such that the cube of the third is the sum of the cubes of the first two.

   Solution: Let the three consecutive integers be x, x + 1 and x + 2. Then we know that:

   $(x+2)^3 = x^3 + (x+1)^3$
   $x^3 + 6x^2 + 12x + 8 = x^3 + x^3 + 3x^2 + 3x + 1$
   $x^3 + 6x^2 + 12x + 8 = 2x^3 + 3x^2 + 3x + 1$
   $7 = x^3 – 3x^2 – 9x$
   $7 = x(x^2 – 3x – 9)$

   By the Fundamental Theorem of Arithmetic, x would have to be either 1 or 7. Neither case works (33 ≠ 13 + 23 and 93 ≠ 73 + 83). Therefore there are no three consecutive positive integers for which this is true

6. (8 points) The Department of Homeland Security has commissioned several studies in an effort to develop their own version of gaydar. So far they have determined that 4% of the men in San Francisco are gay, 90% of the gay men in San Francisco say "That's fierce" regularly, and 98% of the nongay men in San Francisco don't say "That's fierce" regularly

   a. (4 points) What is the probability that a man from San Francisco is gay given that he says "That's fierce" regularly?

      Solution: Let E be the event that a man says "That's fierce" and let F be the event that a man is gay. We are given:

$$P(F) = 0.04, P(\overline{F}) = 0.96$$
$$P(E \mid F) = 0.9, P(\overline{E} \mid F) = 0.1$$
$$P(E \mid \overline{F}) = 0.02, P(\overline{E} \mid \overline{F}) = 0.98$$

      From Baye's Rule we know that:

$$P(F \mid E) = \frac{P(E \mid F)P(F)}{P(E \mid F)P(F) + P(E \mid \overline{F})P(\overline{F})} = \frac{0.9 \cdot 0.04}{0.9 \cdot .04 + 0.02 \cdot 0.96} \cong 0.65$$

   b. (4 points) What is the probability that a man from San Francisco is not gay given that he doesn't say "That's fierce" regularly?

      Solution: We have the same values as above and again using Baye's Rule:

$$P(\overline{F} \mid \overline{E}) = \frac{P(\overline{E} \mid \overline{F})P(\overline{F})}{P(\overline{E} \mid \overline{F})P(\overline{F}) + P(\overline{E} \mid F)P(F)} = \frac{0.98 \cdot 0.96}{0.98 \cdot .96 + 0.1 \cdot 0.04} \cong 0.996$$

7. (8 points)

   a. (4 points) Suppose that a fair coin is flipped 20 times and we count how many times we get heads versus tails. What is the probability that the number of heads is not equal to the number of tails?

      Solution: This is a Bernoulli trial where the probability of success (p) is ½ and the probability of failure (q) is ½. The number of heads and tails are equal only for the term that involves $p^{10}q^{10}$, so the probability of that *not* happening is:

$$1 - \binom{20}{10}\frac{1}{2^{20}}$$

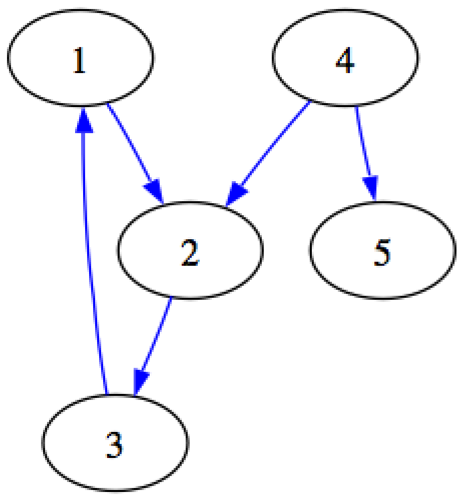
   b. (4 points) Players are allowed to bet on the outcome of the flipping of a fair coin. The coin is flipped four different times. If two of the four flips come up heads, each player wins $1. If three of the four flips come up heads, each player wins $2. If all four flips come up heads, each player wins $4. In all other cases the players win nothing. What is the expected amount won, on average, each time this game is played? Notice that you aren't asked to take into account how much it costs to play this game, just how much a player can expect to win on average.

Solution: There are 16 outcomes possible for flipping the coin 4 times. One of those is all heads (HHHH), 4 of those have 3 heads (HHHT, HHTH, HTHH, THHH), and 6 of those have 2 heads (HHTT, HTHT, HTTH, THHT, THTH, TTHH). The probabilities are simply the number of outcomes divided by 16 and the expected value is the sum of the product of the amount won times the probability, so the answer is:

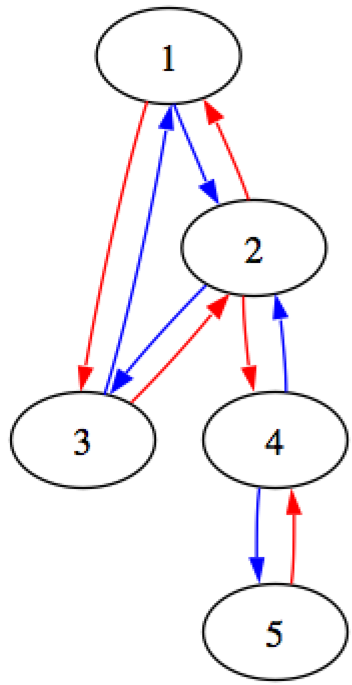
$$(\$4)\left(\frac{1}{16}\right)+(\$2)\left(\frac{4}{16}\right)+(\$1)\left(\frac{6}{16}\right)=\$1.125$$

8. (8 points) Let R be the relation {(1, 2), (2, 3), (3, 1), (4, 2), (4, 5)} defined on the set {1, 2, 3, 4, 5}.

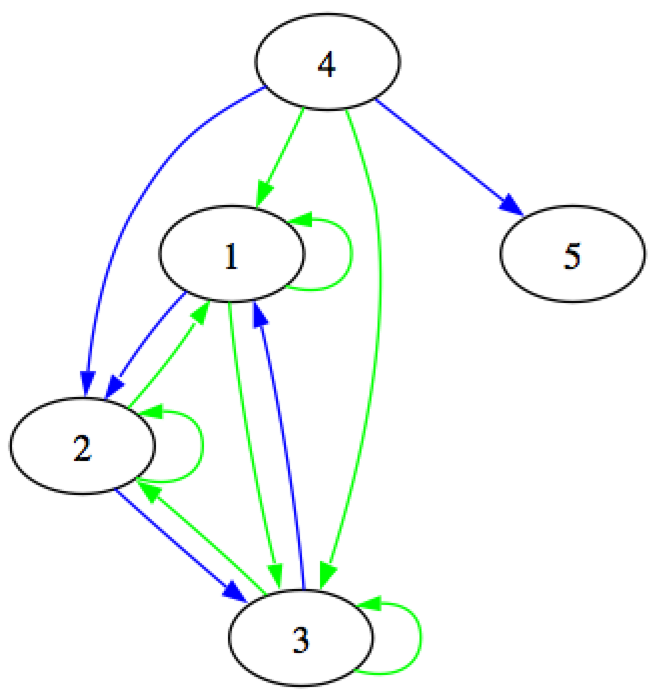
   a. (2 points) Draw the graph of R.



b. (2 points) Draw the graph of the symmetric closure of R.



c. (4 points) Draw the graph of the transitive closure of R.



9. (10 points) Prove or disprove that a number is irrational if its square is irrational.

   Solution: The statement is true. We can use an indirect proof (proof by contrapositive) by showing that if a number is rational, then its square is rational. Let n be a rational number. Then n can be expressed as a/b where a and b are integers and $b \neq 0$, which means n^2 is a^2/b^2. But if a and b are integers then so are a^2 and b^2 and if $b \neq 0$ then $b^2 \neq 0$. Thus, n^2 is also rational, which completes the proof

10. (12 points) Assuming that the sum of two real-valued variables x and y is a constant c:

    $x + y = c$

    Prove that the product xy is maximal when x = y. You are not allowed to use calculus.

    Solution: We prove this by contradiction. Suppose that there is a number c such that x + y = c and the product xy is not maximal when x = y. If x and y were equal, they would each be c/2. If that is not maximal, then there is a number delta such that one of x and y is (c/2 + delta) and the other is (c/2 – delta). The product of x and y has to be greater than when they are equal, so we have that:

$$(c/2 + delta)(c/2 - delta) > (c/2)(c/2)$$
$$c^2/4 - delta^2 > c^2/4$$
$$-delta^2 > 0$$
$$delta^2 < 0$$

    There is no number whose square is negative, so this is a contradiction. Therefore, for every number c, if x + y = c then the product xy is maximal when x and y are equal.

11. (8 points) A university needs to assign 180 students to 3 different dorms called LittleDorm, MediumDorm and BigDorm. LittleDorm houses 40 people, MediumDorm houses 60 people and BigDorm houses 80 people. Of the 180 students to be assigned to dorms, 65 are women and 115 are men.

a. (2 points) How many ways are there to assign the 180 students to the 3 dorms?

Solution: We can choose the 40 students for LittleDorm first and then choose the 60 students for MediumDorm. The remaining students all go to BigDorm, so the answer is:

$$\binom{180}{40}\binom{140}{60}$$

b. (2 points) How many ways are there to assign the 180 students to the 3 dorms if LittleDorm is to be all men?

Solution: We can pick the 40 men for LittleDorm first from the 115 men and then pick the 60 students for MediumDorm from the remaining 140 students, and the rest go to BigDorm, so the answer is:

$$\binom{115}{40}\binom{140}{60}$$

c. (2 points) How many ways are there to assign the 180 students to the 3 dorms if LittleDorm is to be all women and BigDorm is to be all men?

Solution: We can pick the 40 women for LittleDorm from the 65 women and then pick the 80 men for BigDorm from the 115 men and the rest go to MediumDorm, so the answer is:

$$\binom{65}{40}\binom{115}{80}$$

d. (2 points) Suppose that the university has given priority to some students to pick which dorm they want to be in. If 10 students have been allowed to pick LittleDorm, 20 students have been allowed to pick MediumDorm and 25 students have been allowed to pick BigDorm, how many ways are there to assign the remaining students to the 3 dorms?

Solution: Of the 180 students, 55 are already assigned and 125 have yet to be assigned. We can pick the 30 extra students from the 125 to fill LittleDorm, then pick the 40 remaining students needed for MediumDorm from the 95 left, and the rest go to BigDorm, so the answer is:

$$\binom{125}{30}\binom{95}{40}$$