Decimal & Binary Representation Systems

Decimal & binary are positional representation systems

- each position has a value: **`d*base`**$^{\mathtt{i}}$
- for example: $321_{10} = 3*10^2 + 2*10^1 + 1*10^0$
- for example: $101000001_2 =$
  $1*2^8 + 0*2^7 + 1*2^6 + 0*2^5 + 0*2^4 + 0*2^3 + 0*2^2 + 0*2^1 + 1*2^0$

The **general formula** for a positive number in:

- decimal: $\sum_{i=0}^{n} a_i \times 10^{n-i}$, where the a_i are between 0 & 9
- binary: $\sum_{i=0}^{m} b_i \times 2^{m-i}$, where the b_i are 0 or 1

Decimal & Binary Representation Systems

Converting **binary → decimal**:

- evaluate each position & add the factors
- $101000001_2 =$
  $1*2^8 + 0*2^7 + 1*2^6 + 0*2^5 + 0*2^4 + 0*2^3 + 0*2^2 + 0*2^1 + 1*2^0 =$
  $256 + 0 + 64 + 0 + 0 + 0 + 0 + 0 + 1 = 321$

Converting **decimal → binary**:

- decompose the decimal number into powers of 2
- $321 = 256 + 64 + 1 =$
  $1*2^8 + 0*2^7 + 1*2^6 + 0*2^5 + 0*2^4 + 0*2^3 + 0*2^2 + 0*2^1 + 1*2^0 =$
  101000001_2

Hexadecimal Representation System

The hexadecimal numbers:

- `0-9,a,b,c,d,e,f`
- binary values 0000 to 1111
- easier to use than binary numbers (1 digit represents several binary values)
- quick conversion to binary numbers

The general formula for a hexadecimal number is:

- $\sum_{i=0}^{n} a_i \times 16^{n-i}$, where the a_i are between 0 & f
- for example: $141_{16} = 1*16^2 + 4*16^1 + 1*16^0 = 321_{10}$

Converting binary → hexadecimal:

- group into 4-bit numbers: $101001011_2 = 1\ \ 0100\ \ 1011_2$
- translate each group into a hexadecimal digit:
  $1\ \ 0100\ \ 1011_2 = 14B_{16}$

Converting hexadecimal → binary

- expand each hex digit to a sequence of binary digits

Useful Powers of 2

$2^{10} = 1024_{10} \approx 10^3 = 1$ **K**

$2^{20} \approx 10^6 = 1$ **M**

$2^{30} \approx 10^9 = 1$ **G**

Used particularly in storage sizes:

- 16KB cache
- 64MB memory
- 4GB disk

Representing Positive & Negative Numbers

Can represent 2^n different values in `n` bits

For **unsigned integers**, the values are $0..2^{32}-1$

Need a representation for **signed integers** with the following properties:

- an equal number of positive & negative numbers
- a unique representation for 0
- an easy hardware test for 0
- an easy hardware test for the sign
- easy hardware rules for addition/subtraction

Two's Complement

First, some definitions:

- **least significant bit (lsb)**: the least magnitude bit (or digit), the one at the *rightmost* position of the representation
- **most significant bit (msb)**: the greatest magnitude bit (or digit), the one at the *leftmost* position of the representation

Representation for signed integers

- 0 is a series of zeros
- positive numbers: msb = 0
- negative numbers: msb = 1

To represent a negative number:

- start with the representation for its positive value
- flip all the bits (1's to 0; 0's to 1)
- add 1 to the lsb using binary arithmetic

Two's Complement

Examples with a 4-bit binary number:

- What is the representation for 6_{10}?
- What is the representation for -6_{10}?
- What is the representation of 0?
- What is the range of positive numbers?
- What is the range of negative numbers?
- How do you represent 6_{10} in an 8-bit binary number?
- How do you represent -6_{10} in an 8-bit binary number?
- How does the hardware recognize whether a number is positive or negative?
- How does the hardware recognize whether a number is zero?

Addition/Subtraction in Two's Complement

Addition

- do not treat the sign bit specially; perform an addition on all bits
- if add 2 numbers of opposite signs, this will work fine
- if add 2 positive numbers & result "appears" to be negative (msb = 1)
  - → **overflow** (value won't fit in "word size" number of bits)
  - hardware is using the sign bit as a value
  - generates an **exception** (unscheduled procedure call to the operating system) in the program *(we'll discuss exceptions at the end of the quarter)*
- if add 2 negative numbers & result "appears" to be positive (msb = 0)
  - → **underflow**
  - generates an exception in the program

Subtraction

- take the 2's complement of the subtrahend & add it to the other operand

Rules are in Figure 4.4

Alternative Representations

Historically there have been other representations for signed integers, but they are no longer used

Signed magnitude

- separate bit for the sign
- \- extra step to set it
- \- not clear where to store it
- \- has both positive & negative values for zero

One's complement

- negative number is the complement of the absolute value
- \+ positive & negative values are balanced
  - largest positive value: $2{,}147{,}483{,}647_{10}$
  - largest negative value: $-2{,}147{,}483{,}647_{10}$
- \- has 2 values for zero
  - positive zero: 00.....00
  - negative zero: 11.....11

A Bag of Bits

Bit patterns have no meaning

Their meaning depends on how they are interpreted:

- signed integers
- unsigned integers
- floating point numbers
- characters
- instructions

For data, the interpretation is determined by the instruction.