Section 02: Asymptotic Analysis

Section Problems

1. Comparing growth rates

(a) Simplify each of the following functions to a tight big-$\mathcal{O}$ bound in terms of n. Then order them from fastest to slowest in terms of asymptotic growth. (By "fastest", we mean which function increases the most rapidly as n increases.)

- $\log_4(n) + \log_2(n)$
- $\frac{n}{2} + 4$
- $2^n + 3$
- $750,000,000$
- $8n + 4n^2$

(b) Order each of these more esoteric functions from fastest to slowest in terms of asymptotic growth. (By "fastest", we mean which function increases the most rapidly as n increases.) Also state a simplified tight $\mathcal{O}$ bound for each.

- $2^{n/2}$
- 3^n
- 2^n

2. True or false?

(a) In the worst case, finding an element in a sorted array using binary search is $\mathcal{O}(n)$.

(b) In the worst case, finding an element in a sorted array using binary search is $\Omega(n)$.

(c) If a function is in $\Omega(n)$, then it could also be in $\mathcal{O}(n^2)$.

(d) If a function is in $\Theta(n)$, then it could also be in $\mathcal{O}(n^2)$.

(e) If a function is in $\Omega(n)$, then it is always in $\mathcal{O}(n)$.

3. Code to summation

For each of the following code blocks, give a summation that represents the worst-case runtime in terms of n.

(a)

```
int x = 0;
for (int i = 0; i < n; i++) {
    for (int j = 0; j < i; j++) {
        x++;
    }
}
```

(b)

```
int x = 0;
for (int i = n; i >= 1; i /= 2) {
    x += i;
}
```

4. Code modeling

For each of the following code blocks, construct a mathematical function modeling the worst-case runtime of the code in terms of n. Then, give a tight big-$\mathcal{O}$ bound of your model.

(a)

```
int x = 0;
for (int i = 0; i < n; i++) {
    for (int j = 0; j < n * n / 3; j++) {
        x += j;
    }
}
```

(b)

```
int x = 0;
for (int i = n; i >= 0; i -= 1) {
    if (i % 3 == 0) {
        break;
    } else {
        x += n;
    }
}
```

(c)

```
int x = 0;
for (int i = 0; i < n; i++) {
    if (i % 5 == 0) {
        for (int j = 0; j < n; j++) {
            if (i == j) {
                x += i * j;
            }
        }
    }
}
```

(d)

```
int x = 0;
for (int i = 0; i < n; i++) {
    if (n < 100000) {
        for (int j = 0; j < n; j++) {
            x += 1;
        }
    } else {
        x += 1;
    }
}
```

(e)
```
int x = 0;
if (n % 2 == 0) {
    for (int i = 0; i < n * n * n * n; i++) {
        x++;
    }
} else {
    for (int i = 0; i < n * n * n; i++) {
        x++;
    }
}
```

5. Applying definitions

For each of the following, choose a c and n_0 which show $f(n) \in \mathcal{O}(g(n))$. Explain why your values of c and n_0 work.

(a) $f(n) = 3n + 4, g(n) = 5n^2$

(b) $f(n) = 33n^3 + \sqrt{n} - 6, g(n) = 17n^4$

(c) $f(n) = 17\log(n), g(n) = 32n + 2n\log(n)$

6. Using our definitions

Most of the time in the real world, we don't write formal big-$\mathcal{O}$ proofs. The point of having these definitions is not to use them every single time we think about big-$\mathcal{O}$. Instead, we use the formal definitions when a question is particularly tricky, or we want to make a very general statement.

Here are some particularly tricky or general statements that are easier to justify with the formal definitions than with just your intuition.

(a) We almost never say a function is $\mathcal{O}(5n)$, we always say it is $\mathcal{O}(n)$ instead. Show that this transformation is ok, i.e. that if $f(n)$ is $\mathcal{O}(5n)$ then it is $\mathcal{O}(n)$ as well.

(b) When we decide on the big-$\mathcal{O}$ running time of a function, we like to say that whatever happens on small n doesn't matter. Let's see why with an actual proof. You write two functions to solve the same problem: `method1` and `method2`. `method1` takes $\mathcal{O}(n^2)$ time and `method2` takes $\mathcal{O}(n)$ time. What is the big-$\mathcal{O}$ running time of the following function:

```
public void combined(n){
  if(n < 10000)
  method1(n);
  else
  method2(n);
}
```

7. Memory analysis

For each of the following functions, construct a mathematical function modeling the amount of memory used by the algorithm in terms of n. Then, give a **tight** big-$\mathcal{O}$ bound of your model.

(a)

```java
List<Integer> list = new LinkedList<Integer>();
for (int i = 0; i < n * n; i++) {
    list.insert(i);
}
Iterator<Integer> it = list.iterator();
while (it.hasNext()) {
    System.out.println(it.next());
}
```

(b)

```java
int[] arr = {0, 0, 0};
for (int i = 0; i < n; i++) {
    arr[0]++;
}
```

(c)

```java
ArrayDictionary<Integer, String> dict = new ArrayDictionary<>();
for (int i = 0; i < n; i++) {
    String curr = "";
    for (int j = 0; j < i; j++) {
        for (int k = 0; k < j; k++) {
            curr += "?";
        }
    }
    dict.put(i, curr);
}
```

Note 1: For simplicity, assume the dictionary has an internal capacity of exactly n.

Note 2: The amount of memory used by a *single* character (c) and the amount of memory used by a single `int` (x) are both constant.

Note 3: An `ArrayDictionary` stores its key-value pairs contiguously, and performs scans through (potentially) the entire data structure when performing an `insert()` or a `find()`.

8. Case and Asymptotic Analysis

(a) What is the BEST case runtime of the following code? What is the case in which it happens?

```java
void foo(int n) {
    int result = 0;
    if (n % 2 == 0) {
        for (int i = 0; i < n; i++) {
            for (int j = 0; j < i; j++) {
                result++;
            }
        }
    } else {
        for (int i = 1; i < n; i*=2) {
            result++;
        }
    }
    return result;
}
```

(b) What is the Big-Theta runtime of the code above? (Hint: What are the Big-O and Big-Omega runtimes?)

(c) What is the WORST case runtime for the following function? In what case?

```
public class ArrayIntList {
    private int size;
    private int arr;

    public void foo(int val) {
        size++;
        if (size >= arr.length) {
            temp = new int[arr.length * 2];
            for (int i = 0; i < size; i++) {
                temp[i] = arr[i];
            }
            temp[size] = val;
            this.arr = temp;
        } else {
            arr[size] = val;
        }
    }
}
```

(d) What is the WORST case Big-O runtime? Give only the TIGHTEST Big-O bound possible.

```
// assume 0 <= n < arr.length
int foo(int n, int[] arr) {
    if (n < 1000) {
        for (int i = 0; i < n * n; i += 4) {
            arr[i] = n;
        }
    } else if (n > 50000) {
        arr[n] = 65;
    } else {
        int x = 1000000
        while (x > 50) {
            x /= 2;
        }
    }
    return arr[n];
}
```

(e) What is the BEST case runtime for this problem? What is the WORST case?

```
// n == arr.length
boolean find(int[]arr, int n, int k) {
    for(int i=0; i < n; ++i)
        if(arr[i] == k) return true;
    return false;
}
```