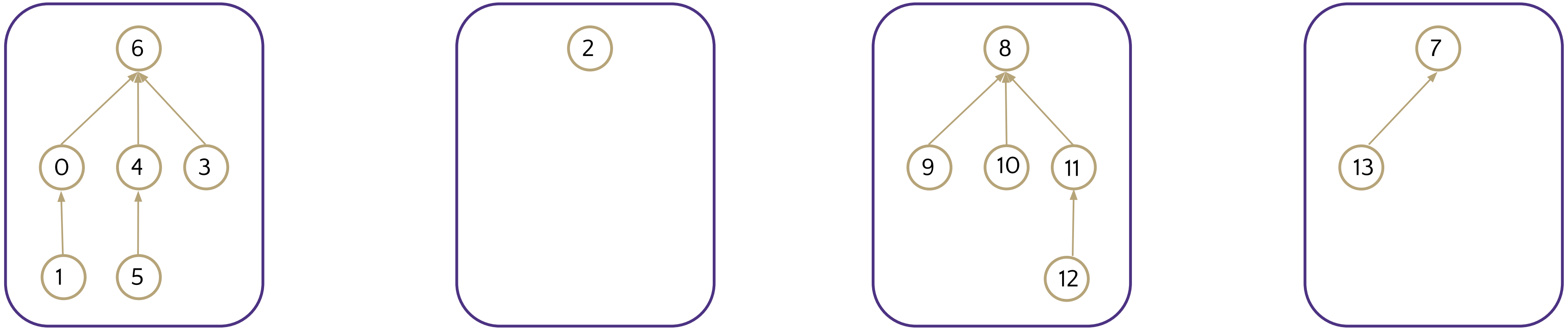
Lecture 20: Disjoint Sets

CSE 373: Data Structures and Algorithms

Practice

Slido Event #1836731
https://app.sli.do/event/nVugNZKCMvb4k9y88C1QaU

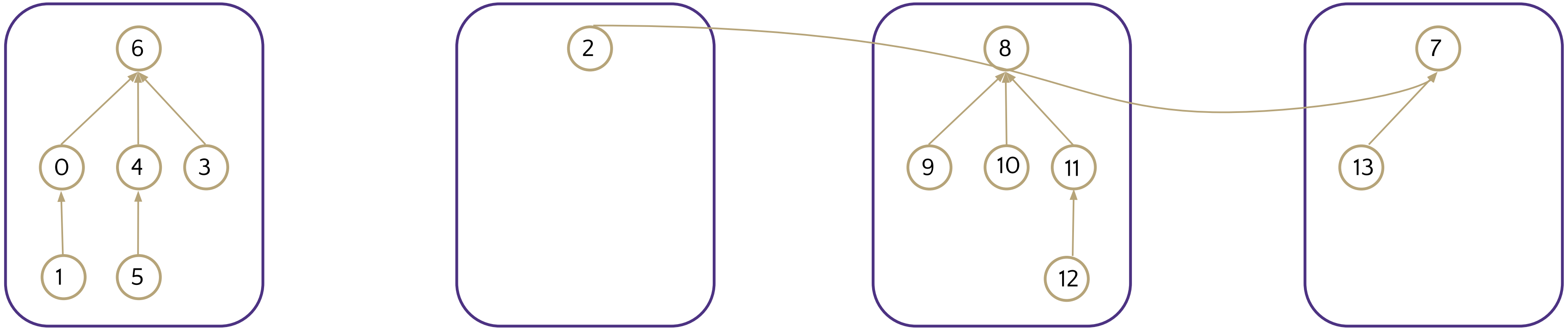
Given the following disjoint-set what would be the result of the following calls on union if we always add the smaller tree (fewer nodes) into the larger tree (more nodes). Draw the forest at each stage with corresponding ranks for each tree.




```
union(2, 13)
union(4, 12)
union(2, 8)
```

Practice

Given the following disjoint-set what would be the result of the following calls on union if we add the "union-by-weight" optimization. Draw the forest at each stage with corresponding ranks for each tree.

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Practice

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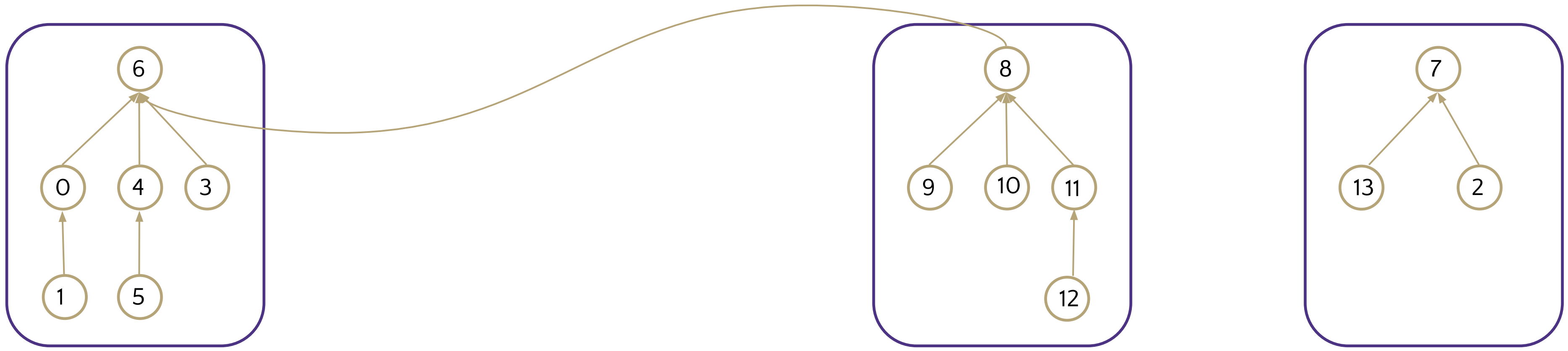


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Practice

Given the following disjoint-set what would be the result of the following calls on union if we add the "union-by-weight" optimization. Draw the forest at each stage with corresponding ranks for each tree.

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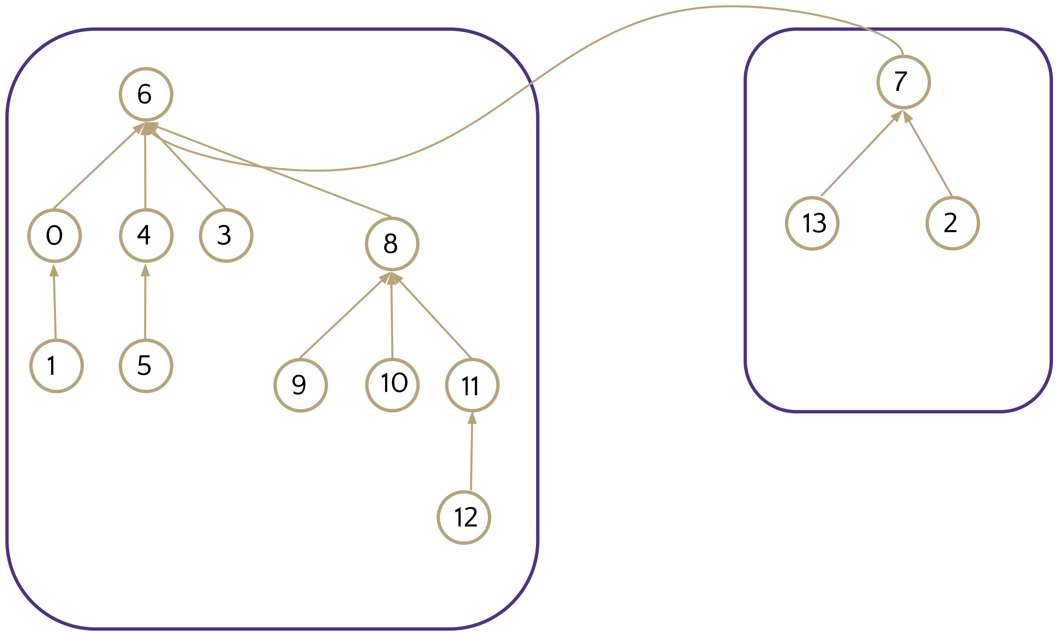

Practice

Given the following disjoint-set what would be the result of the following calls on union if we add the “union-by-weight” optimization. Draw the forest at each stage with corresponding ranks for each tree.

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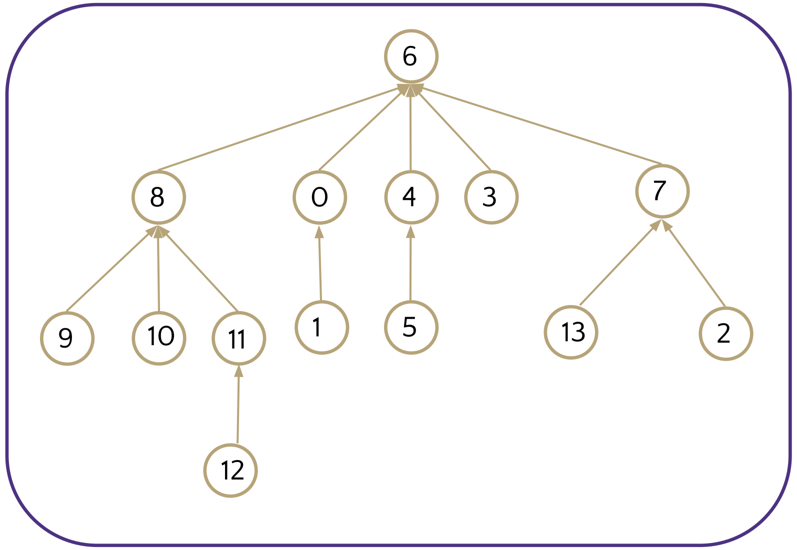


Does this improve the worst case runtimes?

findSet is more likely to be O(log(n)) than O(n)

Announcements

- P4 releases today
  - Due Wednesday 6/7 (finals week)
- EX3 regrade requests due Sunday
- EX5 due Monday
- EX6 releases Monday
- Sorry about lecture audio issues…
  - We are posting the lectures from last year
  - I added a video walk through of Bellman Ford (missing from last year)
  - I recorded a shorter video just going over the Dijkstra's implementation slides to help you on P4

Disjoint Set Implementation

Weighted Union

Path Compression

Array Implementation

New ADT

Set ADT

state

Set of elements
- Elements must be unique!
- No required order

Count of Elements

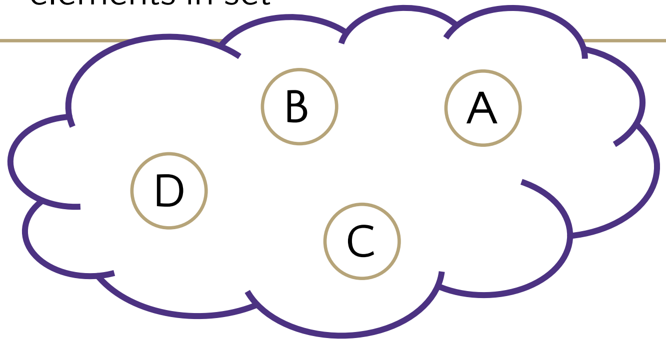
behavior

create(x) – creates a new set with a single member, x

add(x) – adds x into set if it is unique, otherwise add is ignored

remove(x) – removes x from set

size() – returns current number of elements in set



Disjoint-Set ADT

state

Set of Sets
- **Disjoint:** Elements must be unique across sets
- No required order
- Each set has representative

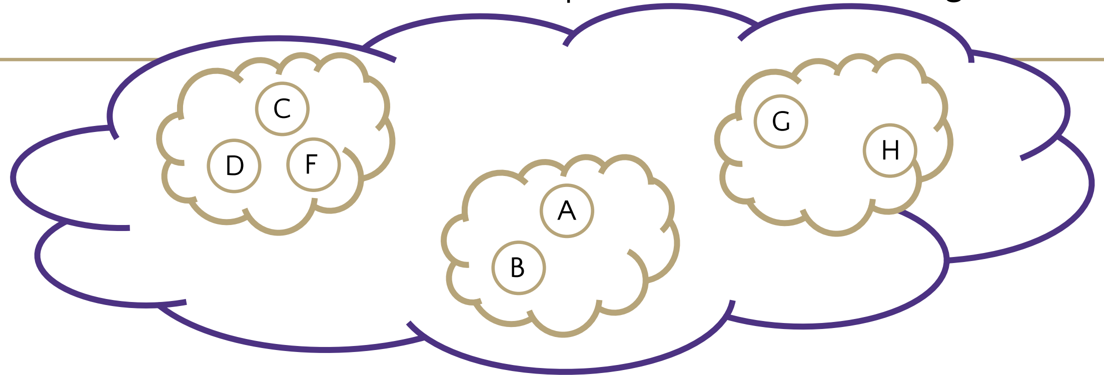
Count of Sets

behavior

makeSet(x) – creates a new set within the disjoint set where the only member is x. Picks representative for set

findSet(x) – looks up the set containing element x, returns representative of that set

union(x, y) – looks up set containing x and set containing y, combines two sets into one. Picks new representative for resulting set



Implementation

Disjoint-Set ADT

state

Set of Sets

- **Disjoint:** Elements must be unique across sets
- No required order
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TreeDisjointSet<E>

state

```
Set<TreeSet> forest
Map<NodeValues,
NodeLocations>
nodeInventory
```

behavior

```
makeSet(x)-create a new
tree of size 1 and add to
our forest

findSet(x)-locates node with
x and moves up tree to find
root

union(x, y)-append tree
with y as a child of tree
with x
```

TreeSet<E>

state

```
SetNode overallRoot
```

behavior

```
TreeSet(x)
add(x)
remove(x, y)
getRep()- returns data of
overallRoot
```

SetNode<E>

state

```
E data
SetNode<E> parent
```

behavior

```
SetNode(x)
updateParent(x)
```

Implementation

Disjoint Sets are built as a collection of three objects
The TreeDisjointSet<E> is the top level object with two fields:

- Set<TreeSet> forest
- Map with node references

Each TreeSet<E> is a collection of SetNodes<E>
Each SetNode<E> can have an unlimited number of children, but only one parent. Nodes store parents instead of children.

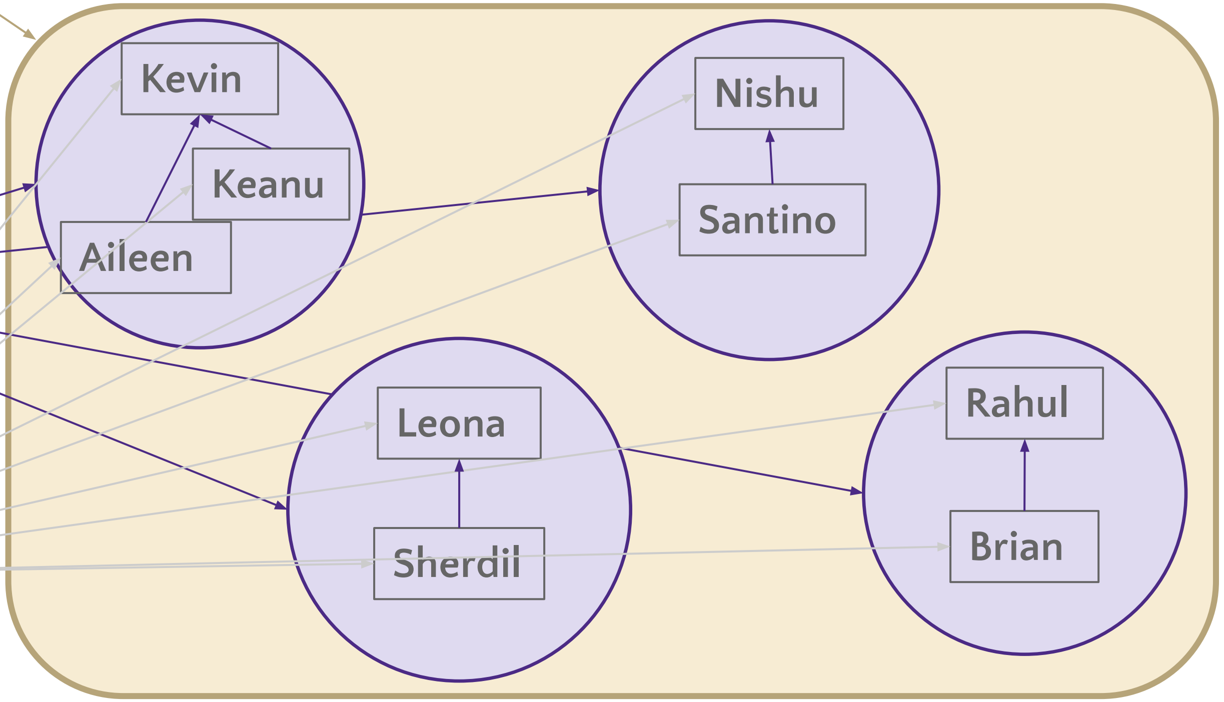
TreeDisjointSet<E> 1

`Set<TreeSet> forest =`

TreeSet<E> 4

`Map<E, SetNode<E>> nodeInventory =`

SetNode<E> 9



Implement makeSet(x)

makeSet(0)

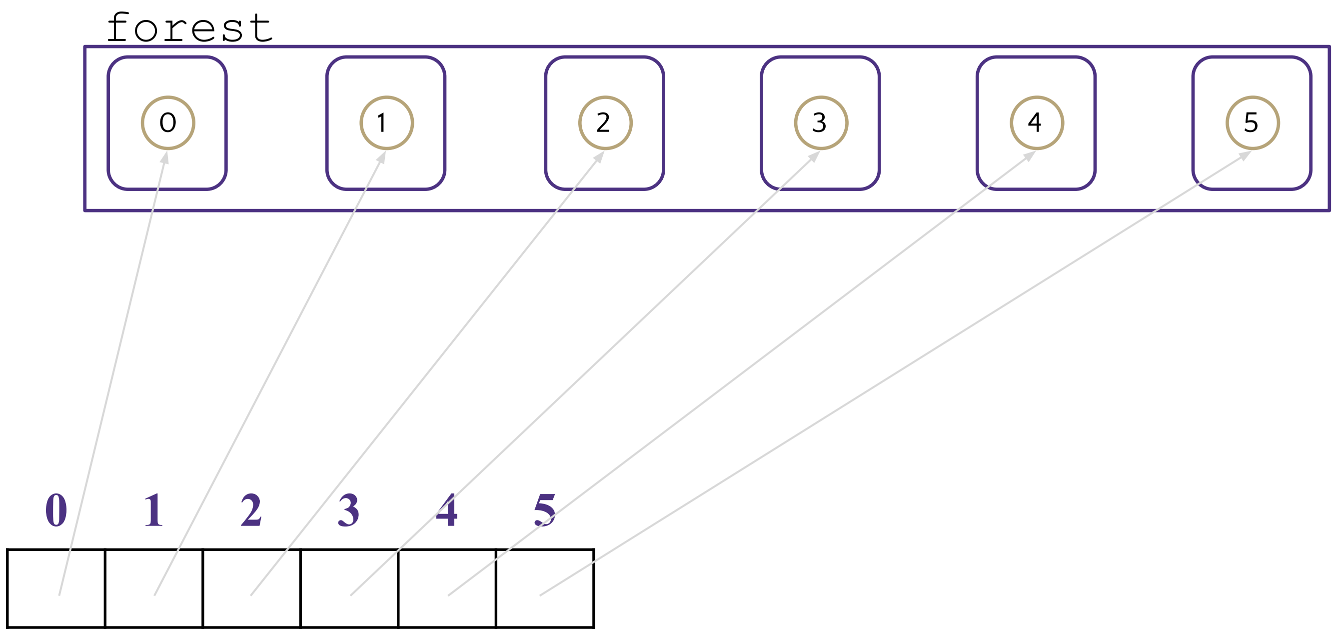
makeSet(1)

makeSet(2)

makeSet(3)

makeSet(4)

makeSet(5)



TreeDisjointSet<E>

state

Collection<TreeSet> forest

Dictionary<NodeValues, NodeLocations> nodeInventory

behavior

makeSet(x)-create a new tree of size 1 and add to our forest

findSet(x)-locates node with x and moves up tree to find root

union(x, y)-append tree with y as a child of tree with x

Worst case runtime?

O(1)

Implement find(X)

```
find(Ken):
  jump to Ken node
  travel upward until root Joyce
  return root “Joyce”
```

```
find(Santino) -> Aileen
find(Ken) -> Joyce
find(Santino) != find(Ken)
find(Santino) == find(Aileen)
```

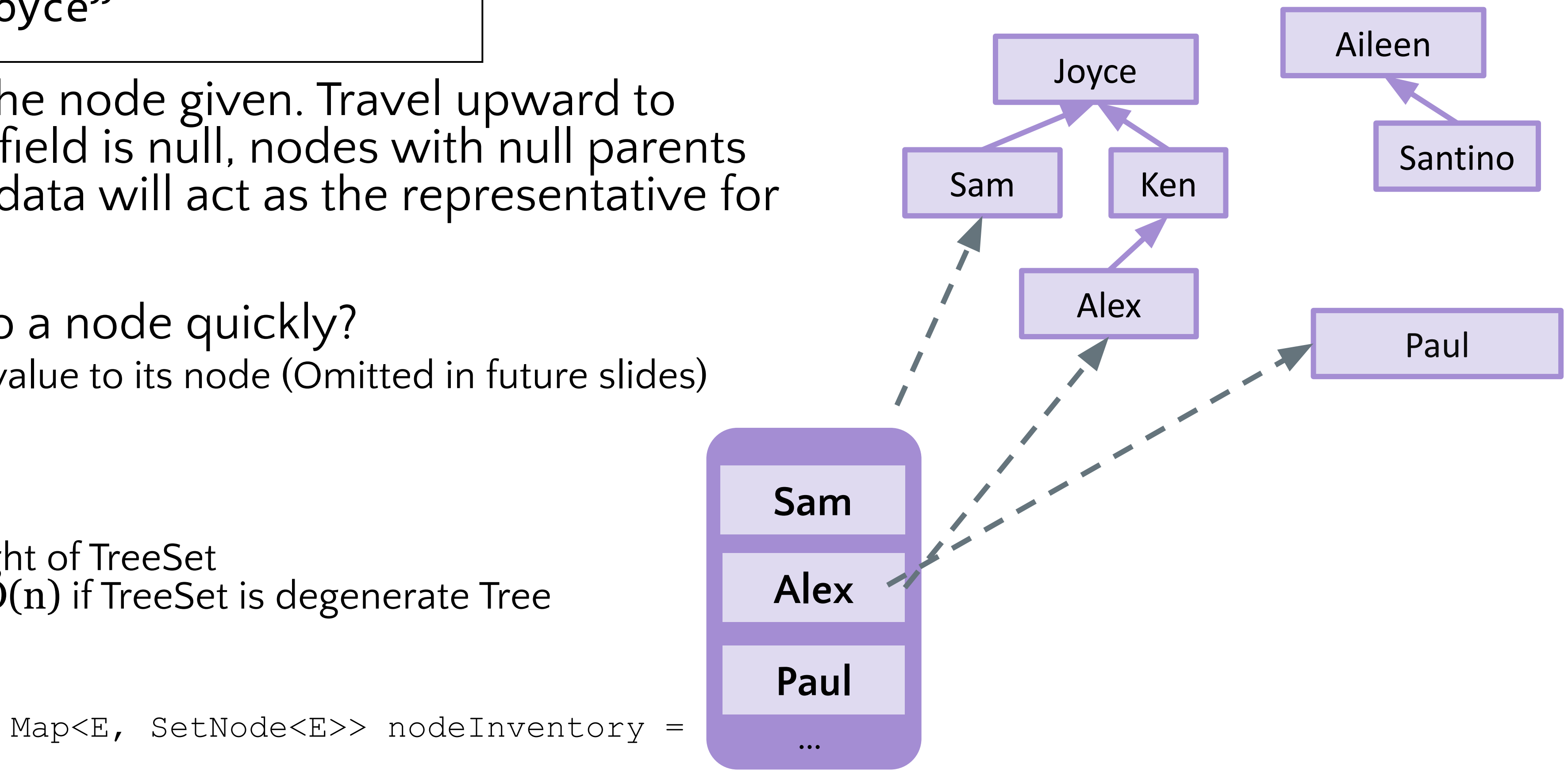
Key Idea: Jump to the node given. Travel upward to parent until parent field is null, nodes with null parents are roots and their data will act as the representative for the set

How do we jump to a node quickly?

- Store a map from value to its node (Omitted in future slides)

Runtime

- jump to node O(1)
- travel up to root
  - based on height of TreeSet
  - Worst case: O(n) if TreeSet is degenerate Tree

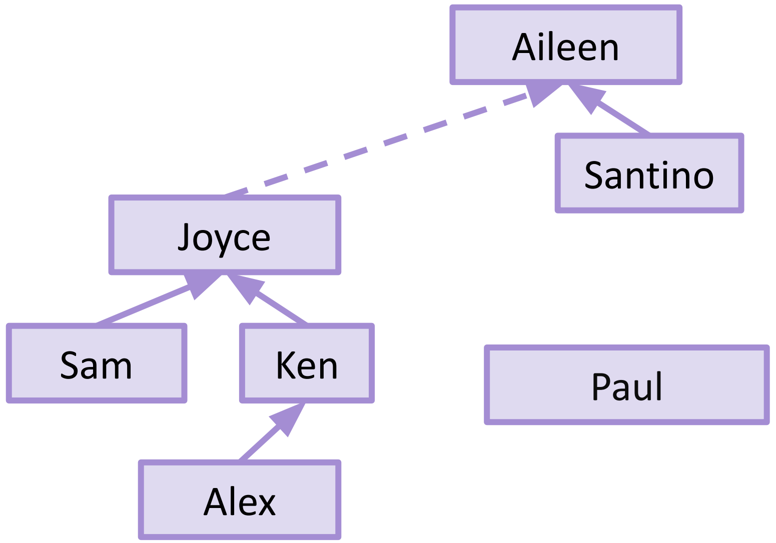


Implement union(x, y)

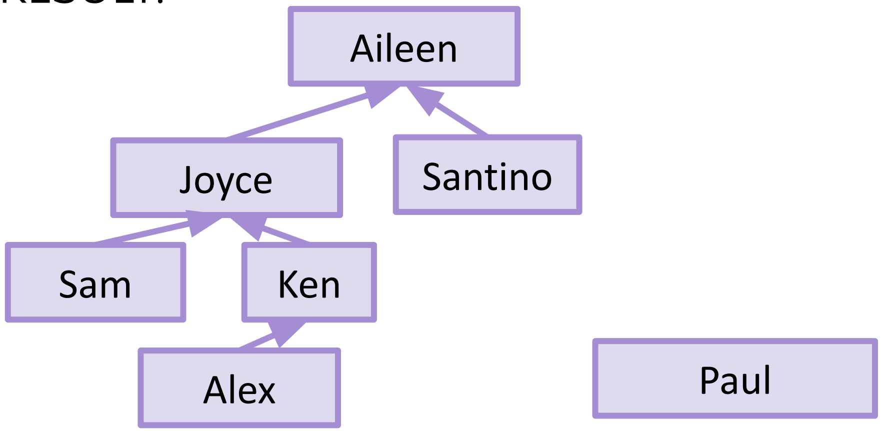
Key idea: easy to simply rearrange pointers to union entire trees together!

- it doesn't matter what the order of the trees are, only that all the nodes from one tree are connected to the other tree

```
union(Ken, Santino):
  rootK = find(Ken) //Joyce
  rootS = find(Santino) //Aileen
  set rootK to point to rootS
```

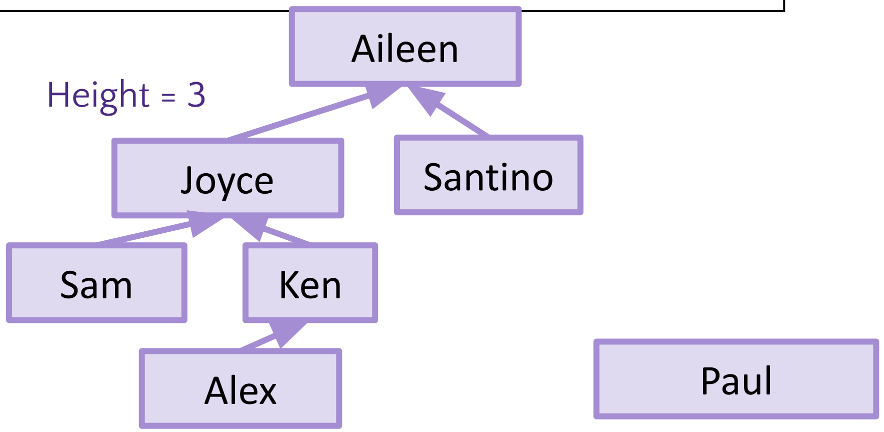


RESULT:



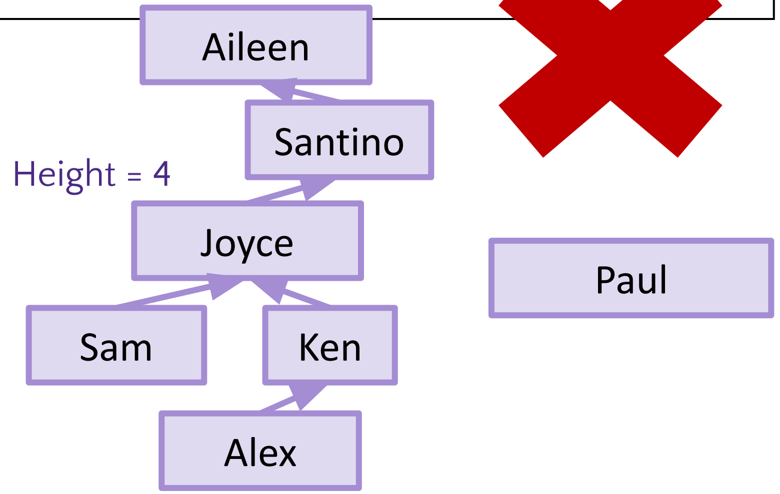
Union: Why bother with the second root?

```
union(Ken, Santino):
  rootK = find(Ken)
  rootS = find(Santino)
  set rootK to point to rootS
```



Why not just use:

```
union(Ken, Santino):
  rootK = find(Ken)
  set rootK to point to Santino
```

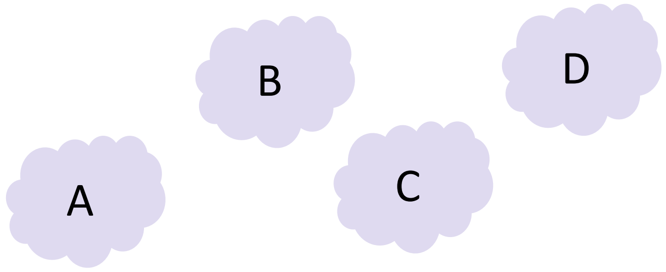


Key idea: keeping the height of each tree short will help minimize runtime for future `find(x)`

- Pointing directly to the individual element instead of the root can grow the tree height

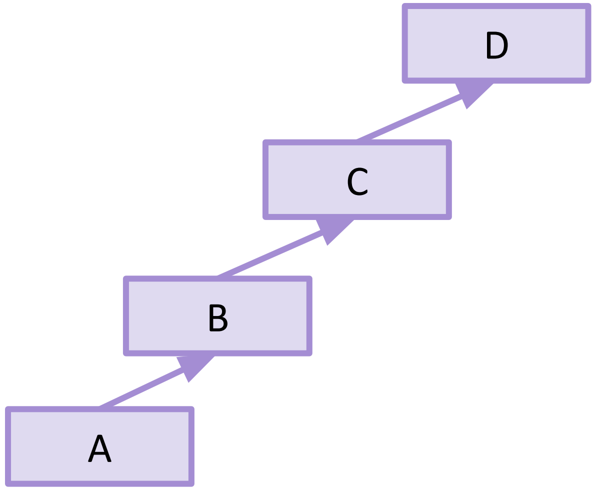
Union runtime

A series of calls to union that would create a worst-case runtime for find on these Disjoint Sets:




```
union(A, B)
union(B, C)
union(C, D)
find(A)
```

n runtime :(



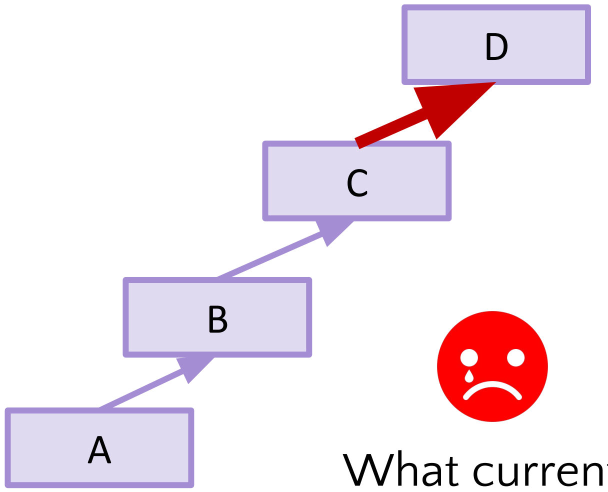

```
find(A):
  jump to A node
  travel upward until root
  return ID
```

```
union(A, B):
  rootA = find(A)
  rootB = find(B)
  set rootA to point to rootB
```

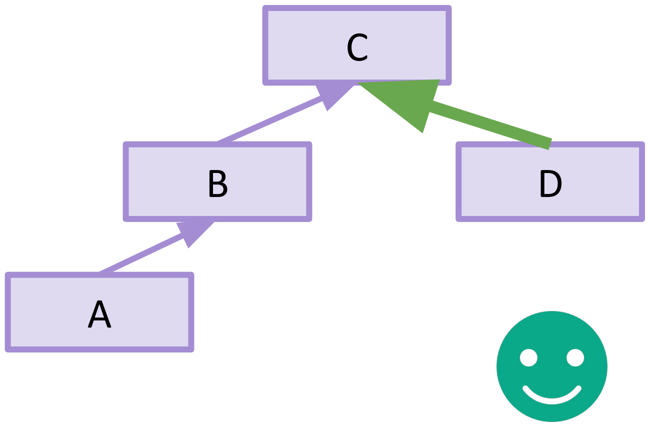
Analyzing the union worst case

- How did we get a degenerate tree?
    - Even though pointing a root to a root usually helps with this, we can still get a degenerate tree **if we put the root of a large tree under the root of a small tree.**
    - Instead of always putting rootA under rootB what if we could ensure the smaller tree went under the larger tree?

`union(C, D)`



What currently happens



What would help avoid degenerate tree

Disjoint Set Implementation

Weighted Union

Path Compression

Array Implementation

WeightedUnion

Goal: Always pick the smaller tree to go under the larger tree

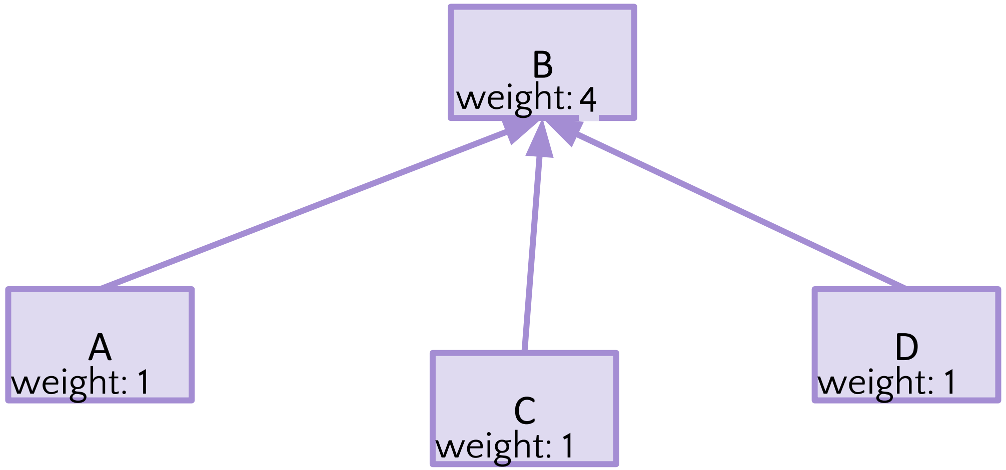
Implementation: Store the number of nodes (or “weight”) of each tree in the root

- Constant-time lookup instead of having to traverse the entire tree to count

```
union(A, B):
  rootA = find(A)
  rootB = find(B)
  put lighter root under heavier root
```

```
union(A, B)
union(B, C)
union(C, D)
find(A)
```

O(1) runtime :)



WeightedUnion: Performance

Consider the worst case where the tree height grows as fast as possible

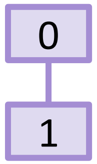
N	H
1	0

0

WeightedUnion: Performance

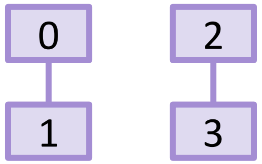
Consider the worst case where the tree height grows as fast as possible

N	H
1	0
2	1



WeightedUnion: Performance

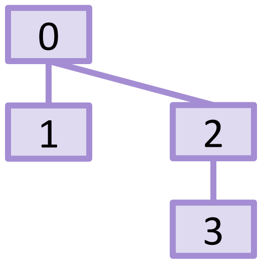
Consider the worst case where the tree height grows as fast as possible



N	H
1	0
2	1
4	?

WeightedUnion: Performance

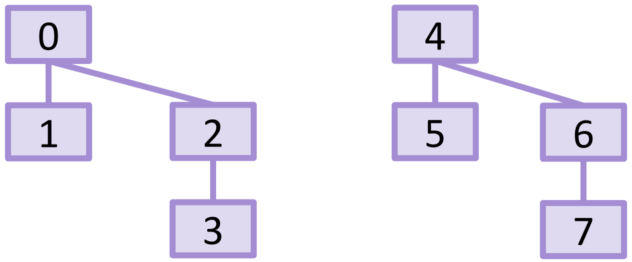
Consider the worst case where the tree height grows as fast as possible



N	H
1	0
2	1
4	2

WeightedUnion: Performance

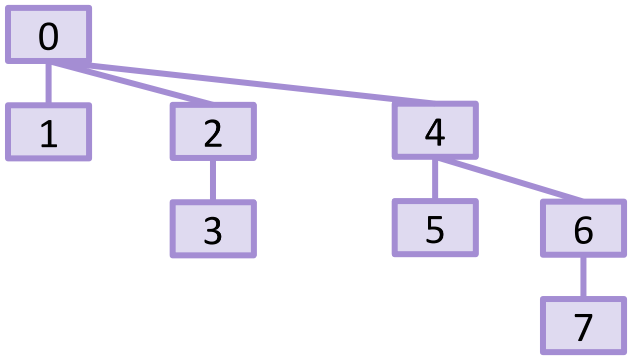
Consider the worst case where the tree height grows as fast as possible



N	H
1	0
2	1
4	2
8	?

WeightedUnion: Performance

Consider the worst case where the tree height grows as fast as possible

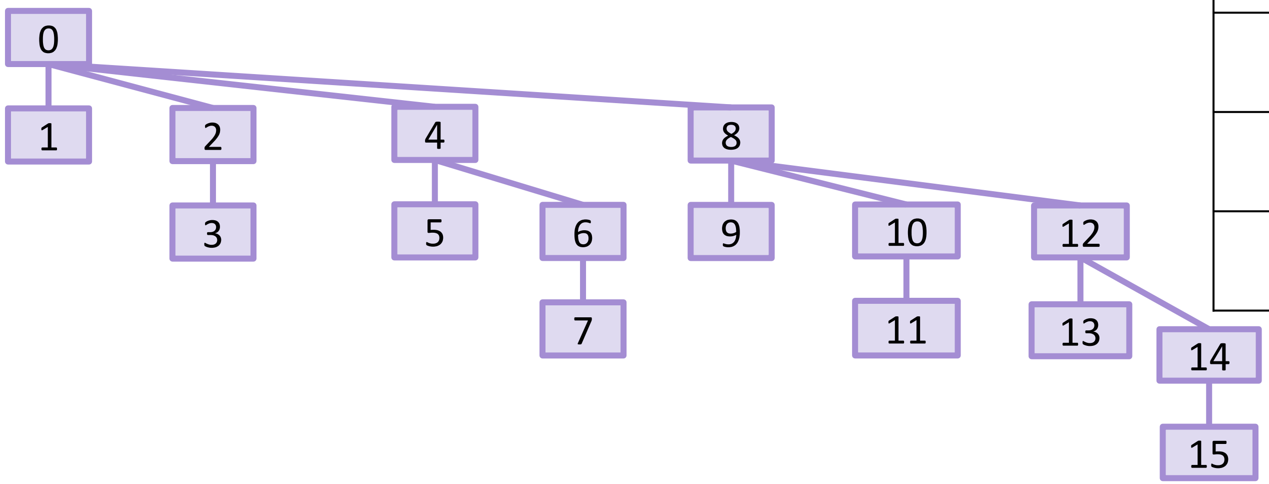


N	H
1	0
2	1
4	2
8	3

WeightedUnion: Performance

Consider the worst case where the tree height grows as fast as possible

Worst case tree height is $O(\log N)$



N	H
1	0
2	1
4	2
8	3
16	4

Runtime so far…

	Worst Case Runtime
`makeSet(value)`	O(1)
`find(value)`	O(log n)
`union(x, y)`	O(log n)

```
kruskalMST(G graph)
  DisjointSets<V> msts; Set finalMST;
  initialize msts with each vertex as single-element MST
  sort all edges by weight (smallest to largest)

  for each edge (u,v) in ascending order:
    uMST = msts.find(u)
    vMST = msts.find(v)
    if (uMST != vMST):
      finalMST.add(edge (u, v))
      msts.union(uMST, vMST);
```

O(ElogV)

This is pretty good! But there's one final optimization we can make:

path compression

Disjoint Set Implementation

Weighted Union

Path Compression

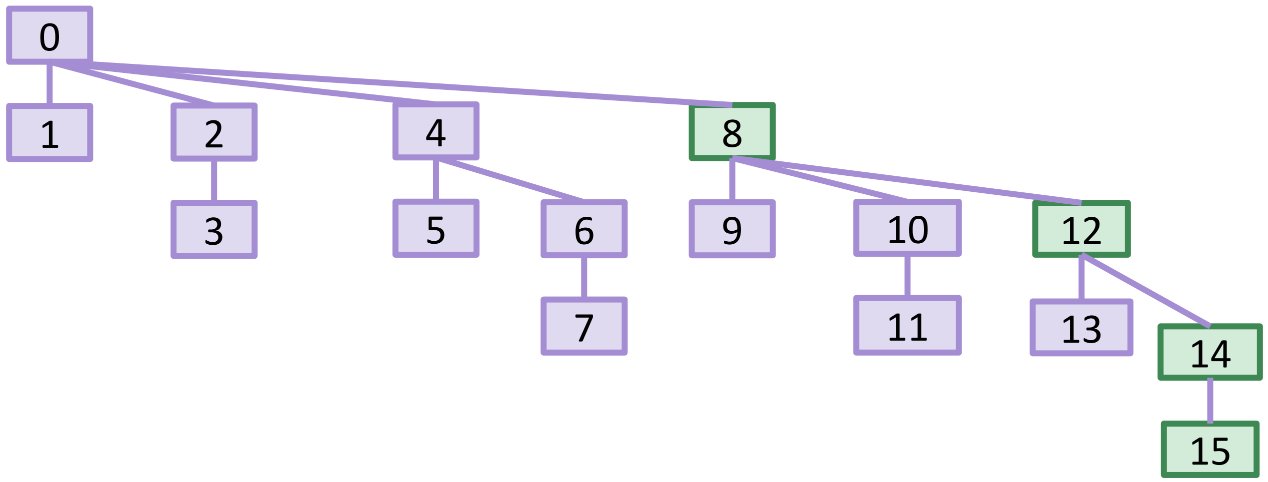
Array Implementation

Modifying Data Structures for Future Gains

- Thus far, the modifications we've studied are designed to *preserve invariants*
  - E.g. Performing rotations to preserve the AVL invariant
  - We rely on those invariants always being true so every call is fast
- Path compression is entirely different: we are modifying the tree structure to *improve future performance*
  - Not adhering to a specific invariant
  - The first call may be slow, but will optimize so future calls can be fast

Path Compression: Idea

This is the worst-case topology if we use WeightedUnion

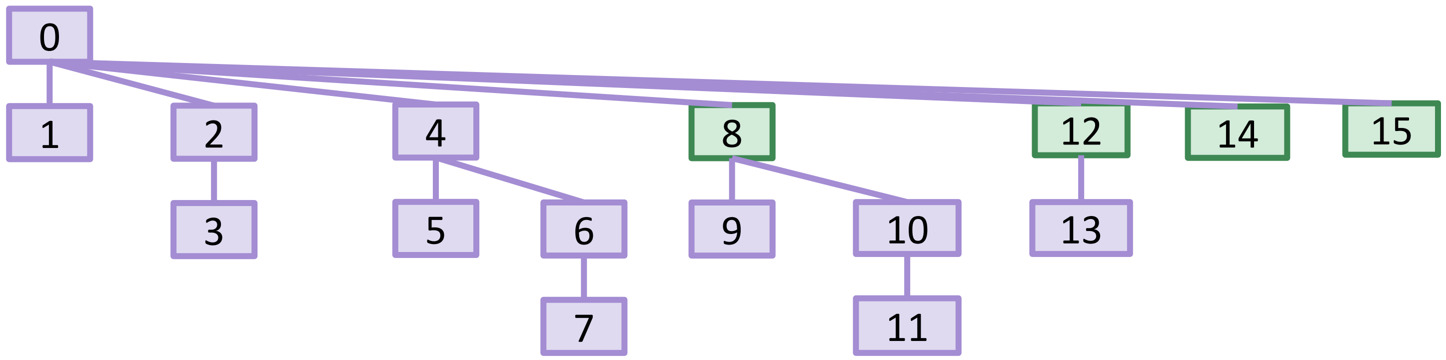


Key Idea: When we do `find(15)`, move all **visited nodes** under the root

- Additional cost is insignificant (we already have to visit those nodes, just constant time work to point to root too)

Path Compression: Idea

This is the worst-case topology if we use WeightedUnion



Key Idea: When we do `find(15)`, move all **visited nodes** under the root

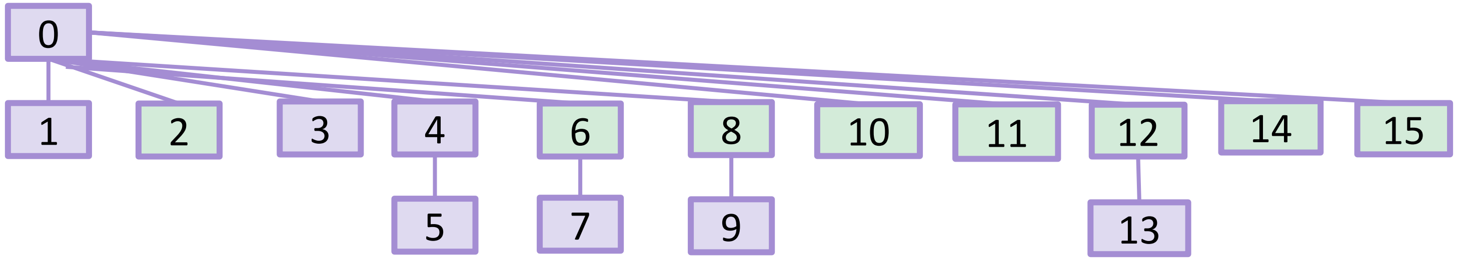
- Additional cost is insignificant (we already have to visit those nodes, just constant time work to point to root too)

Perform Path Compression on every `find()`, so future calls to `find()` are faster!

Path Compression: Details and Runtime

Run path compression on every find()!

- Including the find()s that are invoked as part of a union()

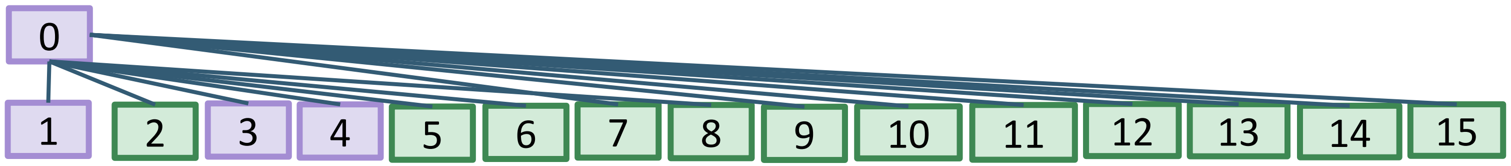


Understanding the performance of M>1 operations requires **amortized analysis**

- Effectively averaging out rare events over many common ones
- Typically used for "In-Practice" case
  - E.g. when we assume an array doesn't resize "in practice", we can do that because the rare resizing calls are *amortized* over many faster calls
- In 373 we don't go in-depth on amortized analysis

Path Compression: Runtime

M `find()`s on WeightedUnion requires takes $O(M \log N)$



… but M `find()`s using the WeightedUnion and PathCompression optimizations takes $O(M \log^* N)$!

- $\log^* n$ is the "iterated log": the number of times you need to apply log to n before it's <= 1
- Note: log* is a loose bound

Path Compression: Runtime

Path compression results in find()s and union()s that are very very close to (amortized) constant time

- log* is less than 5 for any realistic input
- If M find()s/union()s on N nodes is O(M log*N) and log*N ≈ 5, then find()/union()s amortizes to O(1)! 🤯

N	log* N
1	0
2	1
4	2
16	3
65536	4
2^{65536}	5

2^{16}

Number of atoms in the known universe is 2^{256}ish

Kruskal’s Runtime

$$\Theta(|E| \log |E|) \begin{cases} \Theta(|V|) \\ \Theta(|E| \log |E|) \\ \Theta(|E| \log |V|) \begin{cases} \Theta(|E|) \text{ iterations} \\ \Theta(\log |V|) \end{cases} \\ \Theta(|V| \log |V|) \begin{cases} \underline{\text{total}}\ \Theta(|V|) \text{ iterations} \\ \Theta(\log |V|) \end{cases} \end{cases}$$

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kruskalMST(G graph)
  DisjointSets<V> msts; Set finalMST;
  initialize msts with each vertex as single-element MST
  sort all edges by weight (smallest to largest)

  for each edge (u,v) in ascending order:
    uMST = msts.find(u)
    vMST = msts.find(v)
    if (uMST != vMST):
      finalMST.add(edge (u, v))
      msts.union(uMST, vMST)
```

Find and union are log|V| in worst case, but amortized constant “in practice”

Either way, dominated by time to sort the edges ☹️

- For an MST to exist, E can’t be smaller than V, so assume it dominates
- Note: some people write |E|log|V|, which is the same (within a constant factor)

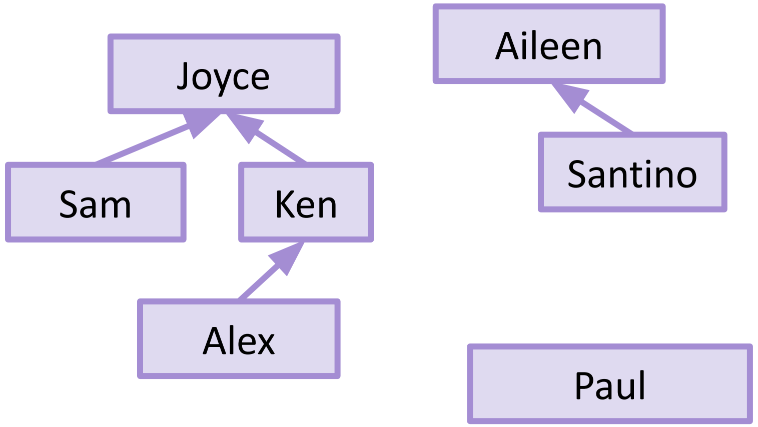
Disjoint Set Implementation

Weighted Union

Path Compression

Array Implementation

Using Arrays for Up-Trees



Since every node can have at most one parent, what if we use an array to store the parent relationships?

Proposal: each node corresponds to an index, where we store the index of the parent (or –1 for roots). Use the root index as the representative ID!

Just like with heaps, tree picture still conceptually correct, but exists in our minds!

0	1	2	3	4	5	6
-1	0	-1	6	-1	2	0
Joyce	*Sam*	*Aileen*	*Alex*	*Paul*	*Santino*	*Ken*

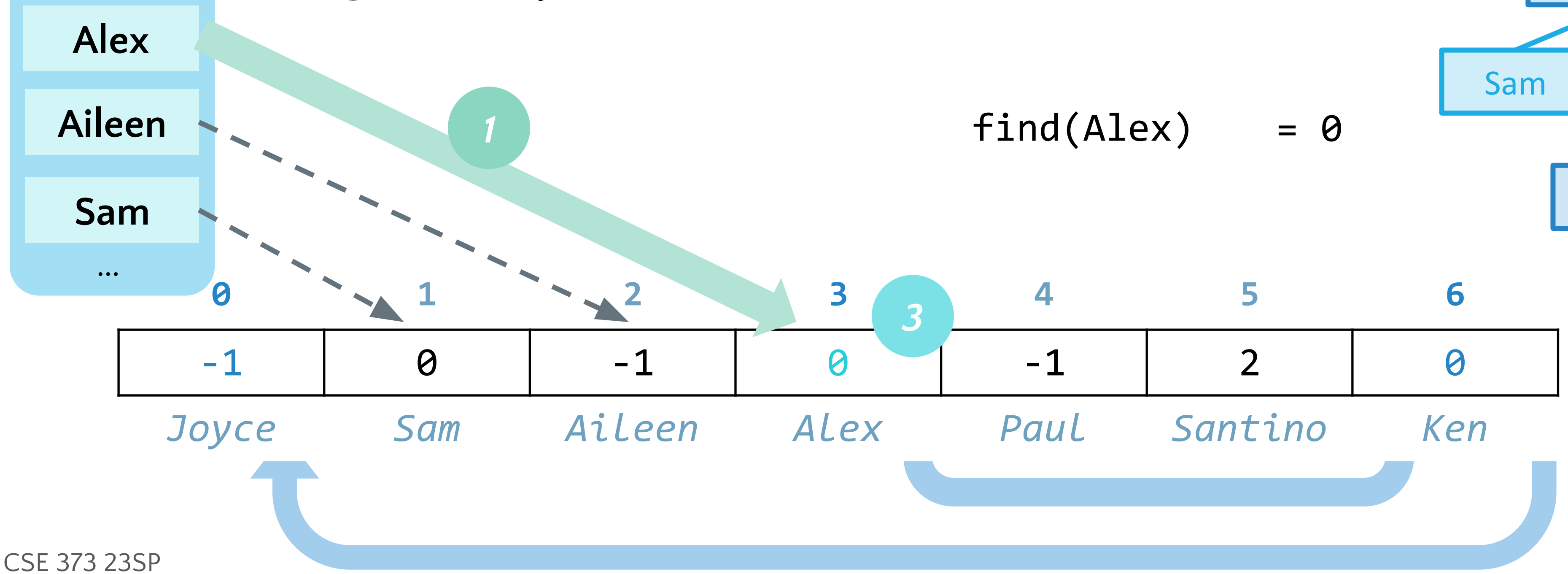
Using Arrays: Find

Initial jump to element still done with extra Map

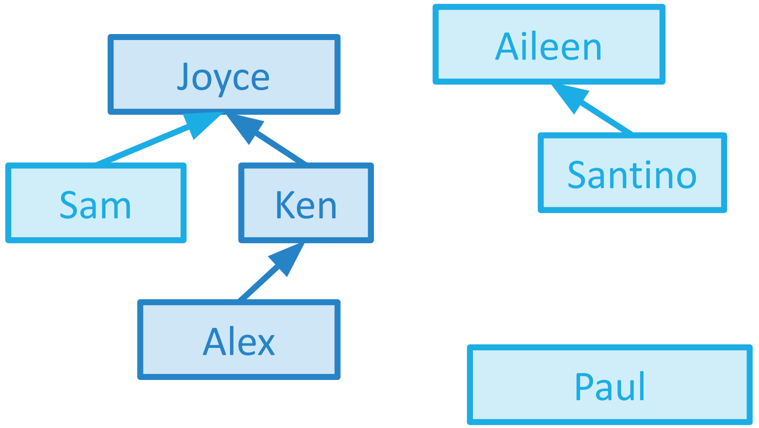
But traversing up the tree can be done purely within the array!

Can still do path compression by setting all indices along the way to the root index!

```
find(A):
  index = jump to A node's index
  while array[index] > 0:
    index = array[index]
  path compression
  return index
```



`find(Alex)    = 0`

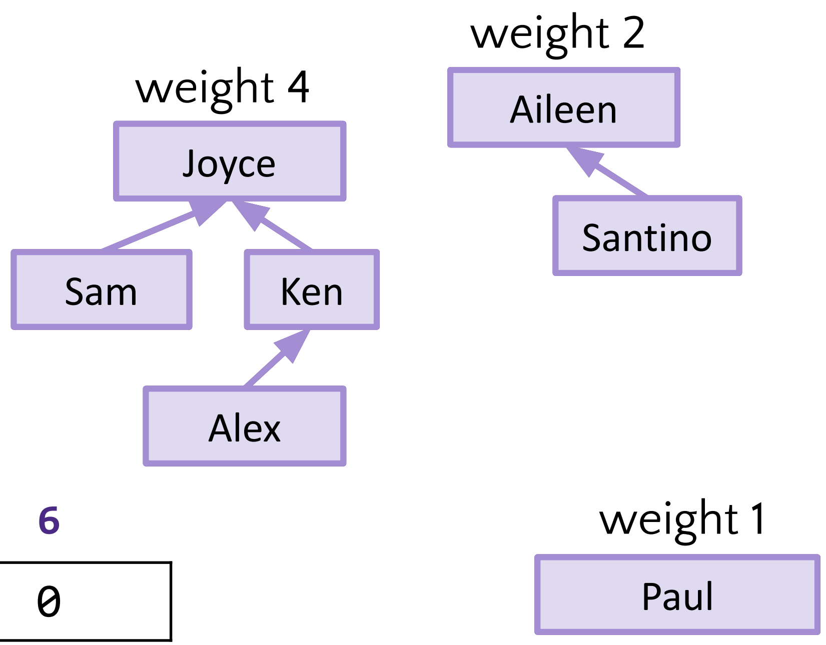


Using Arrays: Union

For WeightedUnion, we need to store the number of nodes in each tree (the weight)

Instead of just storing –1 to indicate a root, we can store –1 * weight!

```
union(A, B):
  rootA = find(A)
  rootB = find(B)
  use -1 * array[rootA] and -1 *
    array[rootB] to determine weights
  put lighter root under heavier root
```



`union(Ken, Santino)`

0	1	2	3	4	5	6
-4	0	-2	6	-1	2	0
Joyce	*Sam*	*Aileen*	*Alex*	*Paul*	*Santino*	*Ken*

Using Arrays: Union

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  put lighter root under heavier root
```
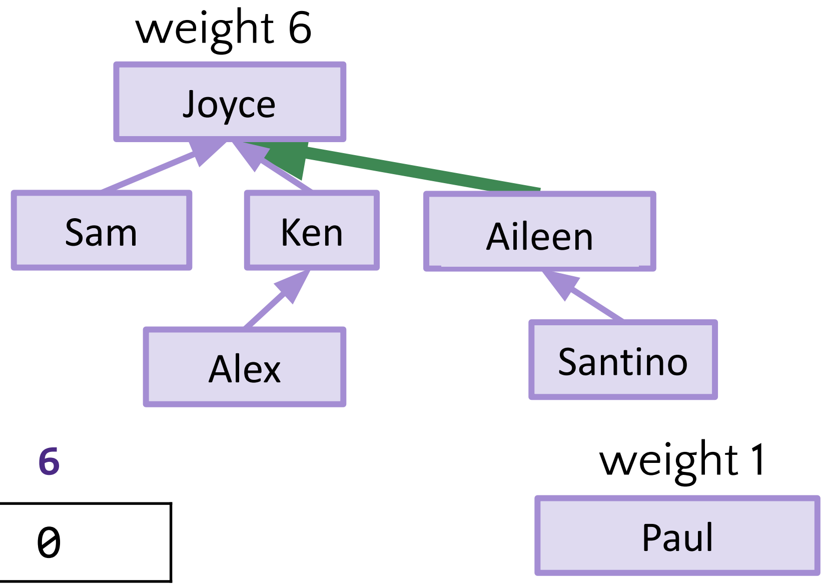
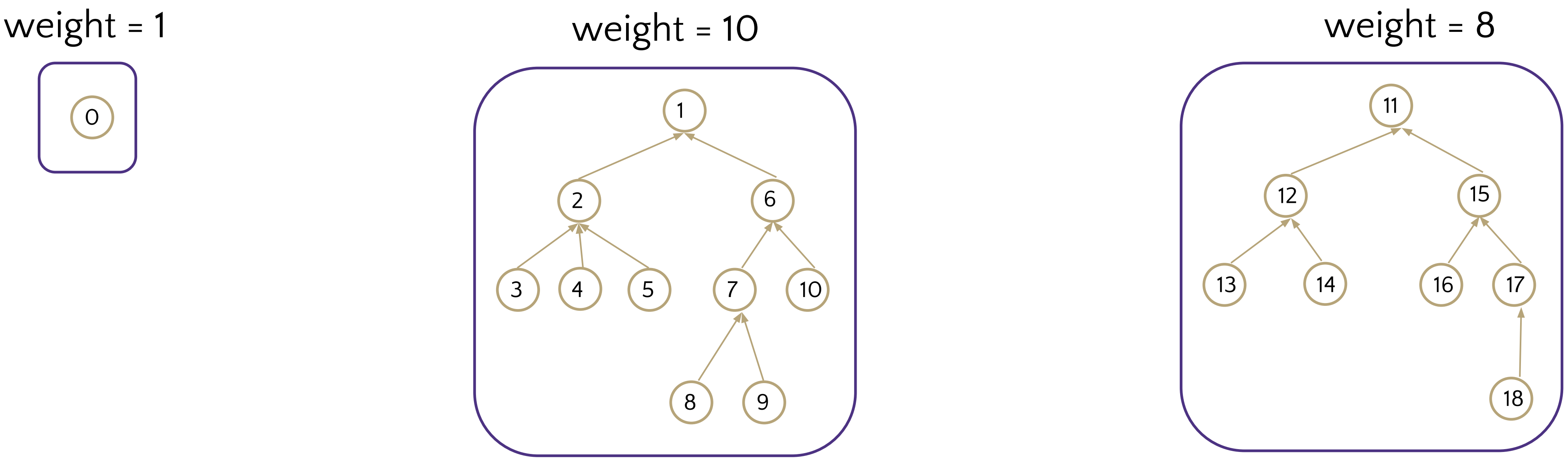



`union(Ken, Santino)`

0	1	2	3	4	5	6
-6	0	0	6	-1	2	0
Joyce	*Sam*	*Aileen*	*Alex*	*Paul*	*Santino*	*Ken*

Array Implementation Practice

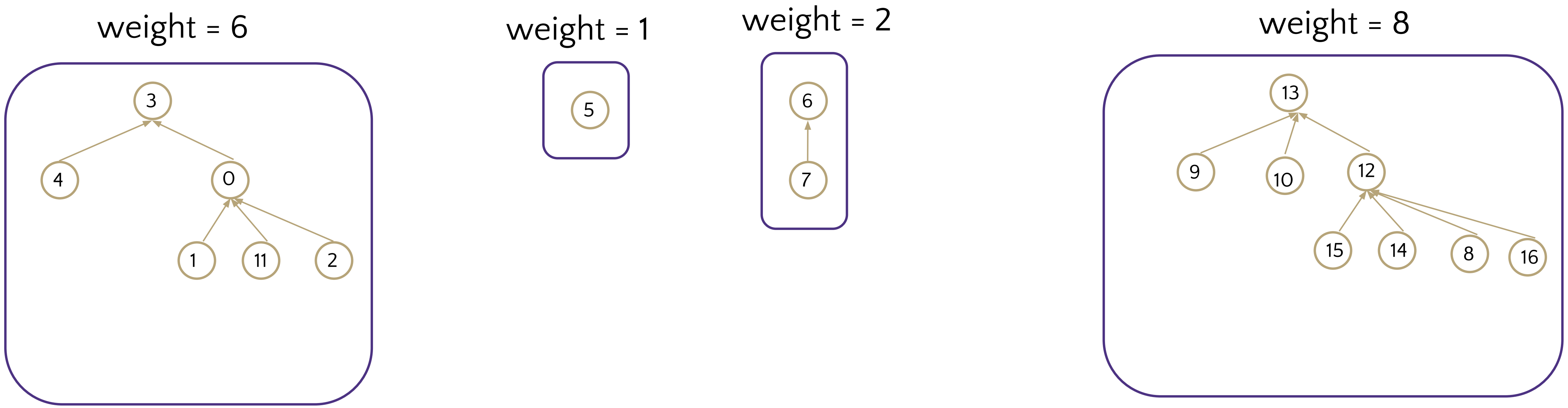


0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
-1	-10	1	2	2	2	1	6	7	7	6	-8	11	12	12	11	15	15	17

Fill in the array representing this DisJoint set. Remember to Store (weight * -1) - 1 as the "parent" of the root nodes

Each "node" now only takes 4 bytes of memory instead of 32

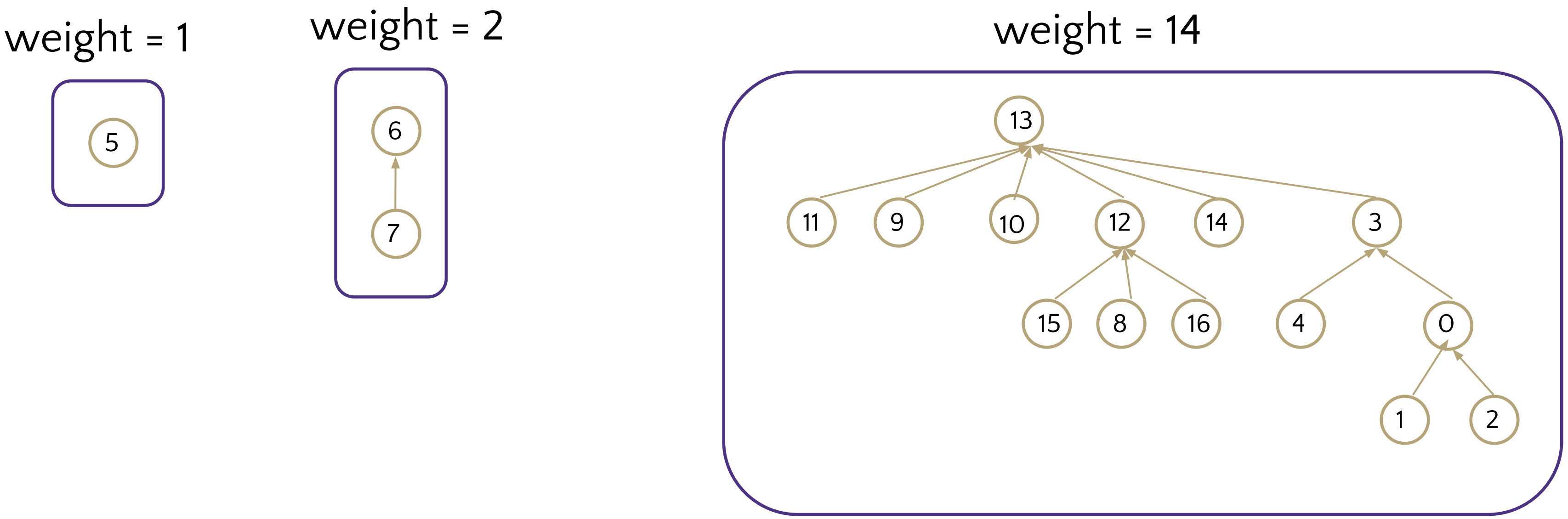
Array Implementation Practice



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3	0	0	-6	3	-1	-2	6	12	13	13	0	13	-8	12	12	12

Update the Array with the correct values after a call of `union(14, 11)` using WeightedUnion and PathCompression

Array Implementation Practice



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3	0	0	13	3	-1	-2	6	12	13	13	13	13	-14	13	12	12

Update the Array with the correct values after a call of `union(14, 11)` using WeightedUnion and PathCompression

Questions?

That's all!

Appendix