Lecture 6: Analyzing Recursive Code

CSE 373: Data Structures and Algorithms

Warm Up

Slido Event #3504442
https://app.sli.do/event/7xnq6sgX28GEKL7yH8PbAz

Which of the following statements are FALSE?

Select all options that apply:

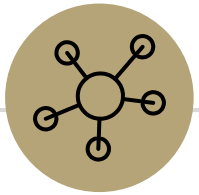
- A Big-Theta bound will exist for every function
- If a function is $O(n^2)$ it can't also be $\Omega(n^2)$
- A piece of code whose model is $f(n) = 3n + 6$ has a simplified tight BigO of $O(n^2)$
- The tight upper and lower bound of piece of code's runtime growth is always the same complexity class

All false!

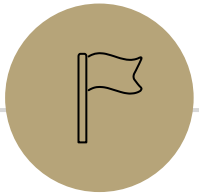
Announcements

- Project 1 is out
  - Due Wednesday April 12th

- Exercise 1 is out
  - Due Monday April 10th



Questions?

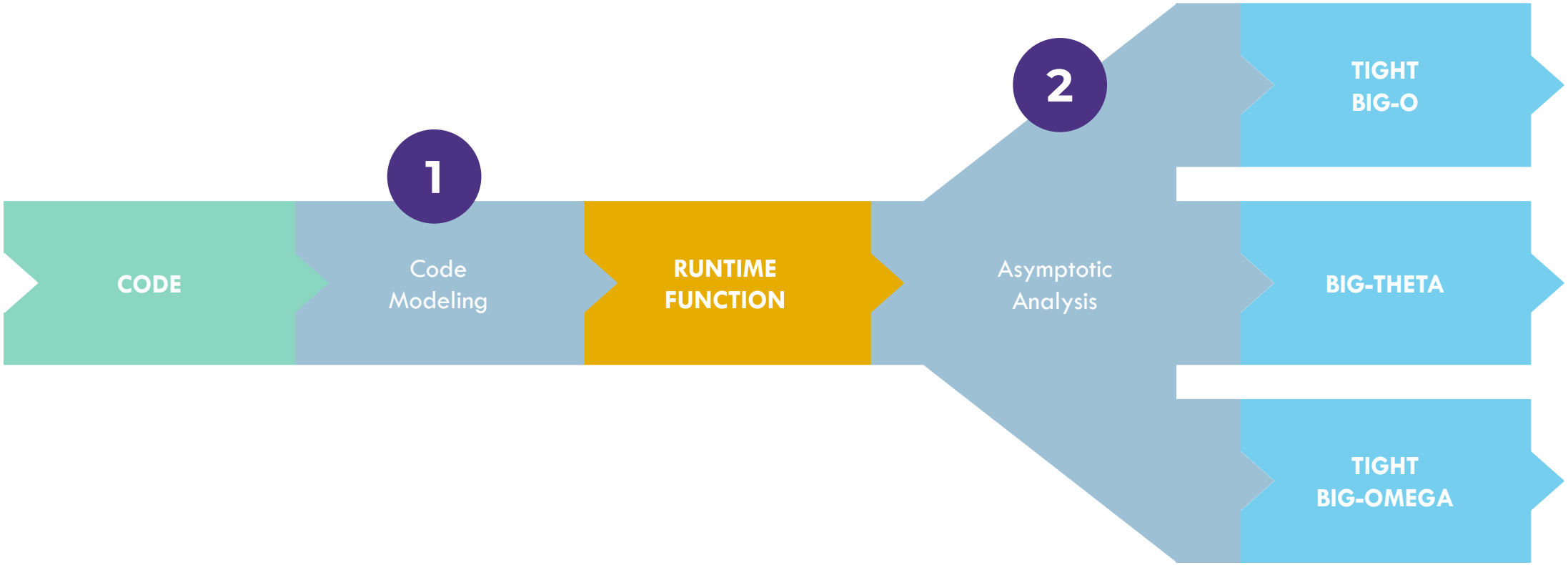


Case Analysis

Modeling Recursive Code

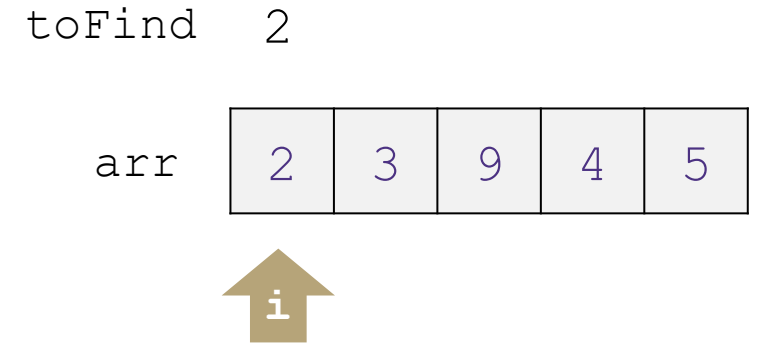
Summations

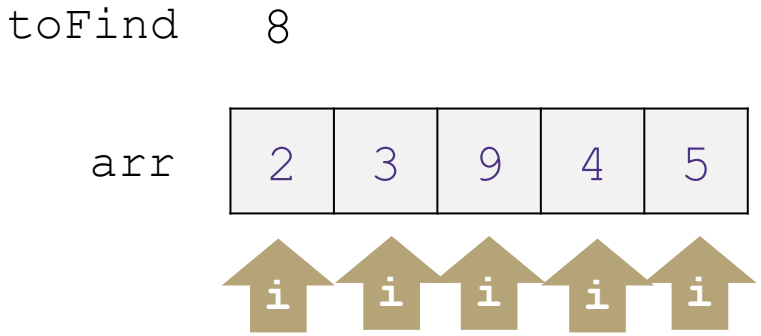
Review: Algorithmic Analysis Roadmap



Case Study: Linear Search

```
int linearSearch(int[] arr, int toFind) {
    for (int i = 0; i < arr.length; i++) {
    if (arr[i] == toFind)
        return i;
    }
    return -1;
}
```





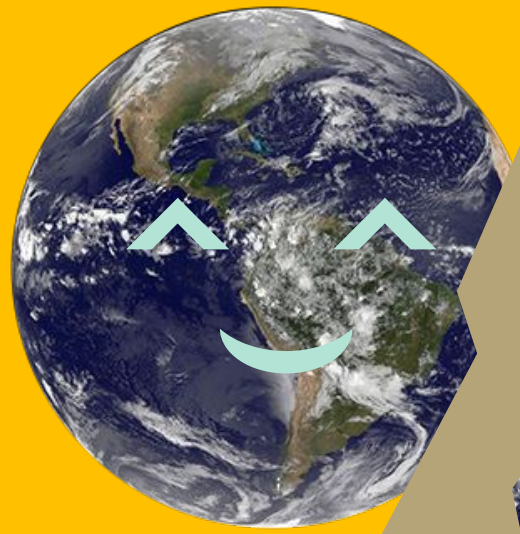
The number of operations doesn't depend just on n.

Even once you fix n (the size of the array) there are still a number of cases to consider.

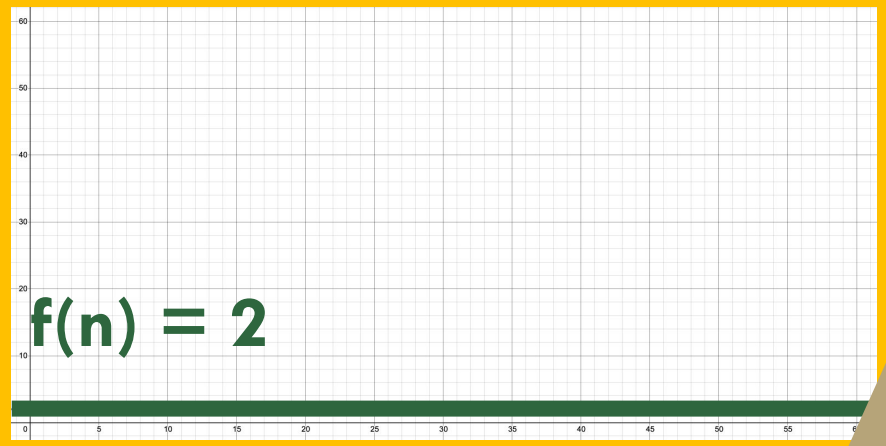
- If `toFind` is in `arr[0]`, we'll only need one iteration, $f(n) = 4$.
- If `toFind` is not in `arr`, we'll need n iterations. $f(n) = 3n + 1$.
- And there are a bunch of cases in-between.

Best Case

On Lucky Earth



```
toFind  2
arr     | 2 | 3 | 9 | 4 | 5 |
          i
```



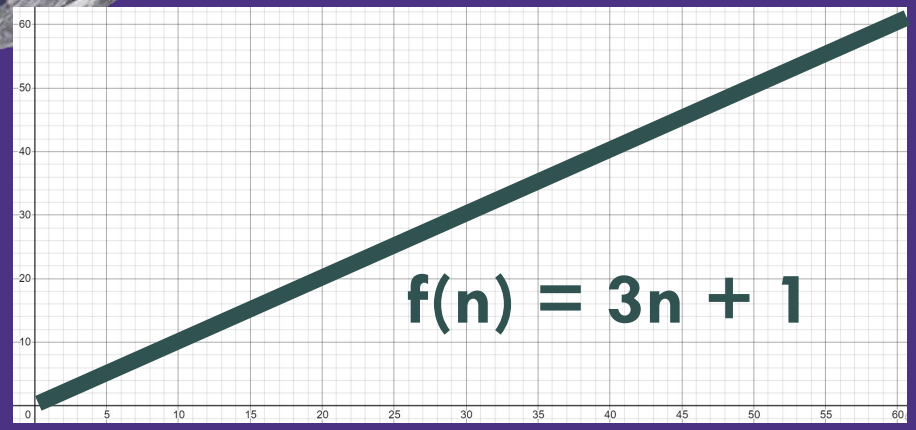
After asymptotic analysis:

O(1) Θ(1) Ω(1)

Worst Case

On Unlucky Earth (where it's 2020 every year)

```
toFind  8
arr     | 2 | 3 | 9 | 4 | 5 |
                      i
```



After asymptotic analysis:

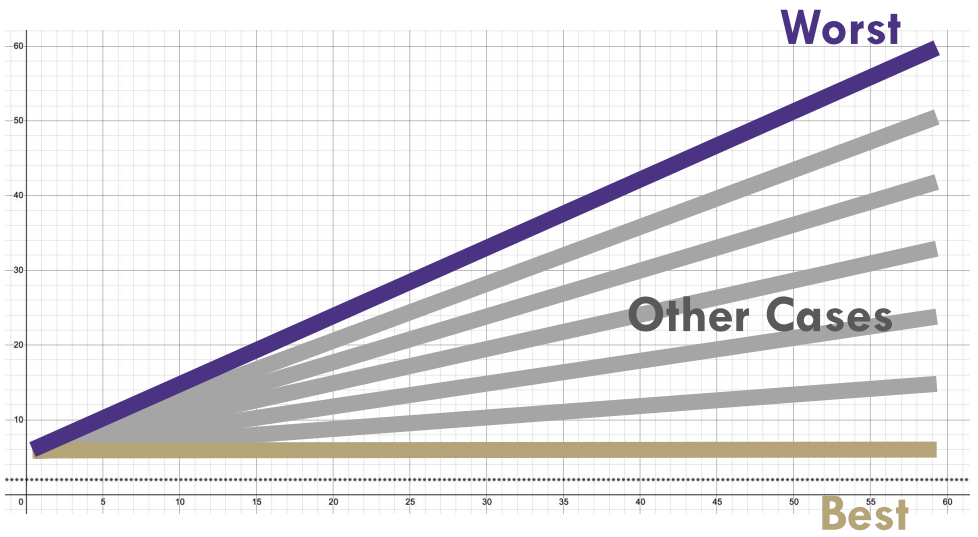
O(n) Θ(n) Ω(n)

Case Analysis

Case: a description of inputs/state for an algorithm that is specific enough to build a code model (runtime function) whose only parameter is the input size

- Case Analysis is our tool for reasoning about **all variation other than n!**
- Occurs during the code □ function step instead of function □ O/Ω/Θ step!

- **Best Case** = fastest that our code could finish on input of size n
- **Worst Case** = slowest that our code could finish on input of size n.
- Importantly, *any* position of `toFind` in `arr` could be its own case!
  - For this simple example, probably don't care (they all still have bound $O(n)$)
  - But intermediate cases will be important later



Caution: Common Misunderstanding

Best/Worst case is based on all variation **other** than value of n

Incorrect – based on specific values of n

- "The best case is when n=1, worst is when n=infinity"
- "The best case is when front is null"
- "The best case is when overallRoot is null"
- "The best case is when n is an even number"

Correct – based on state of data structure regardless of n

- "The best case is when the node I'm looking for is at front, the worst is when it's not in the list"
- "The best case is when the BST is perfectly balanced, the worst is when it's a single line of nodes"

Other cases

"Assume X won't happen" case

- Assuming our array won't need to resize is the most common example

"Average" case

- Assuming your input is random
- Need to specify what the possible inputs are and how likely they are
- ***f*(*n*)** is now the **average** number of steps on a **random** input of size ***n***

"In-practice" case

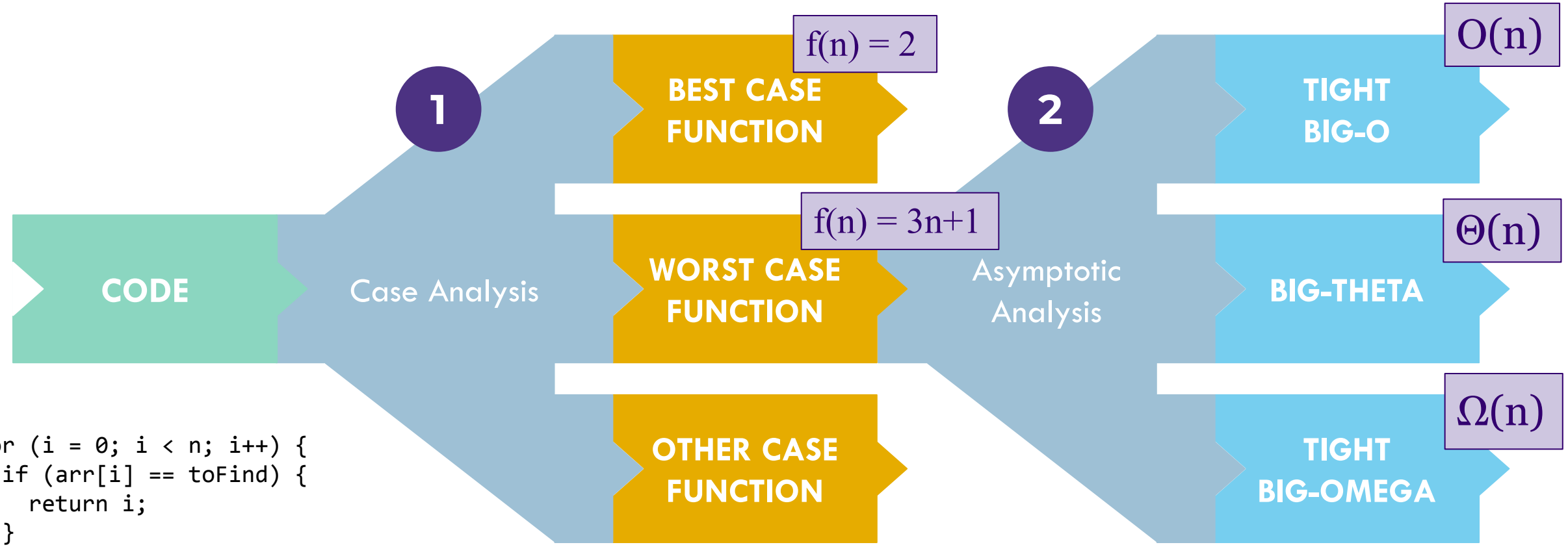
- This isn't a real term (I just made it up)
- Making some reasonable assumptions about how the real-world is probably going to work
- We'll tell you the assumptions and won't ask you to come up with these assumptions on your own
- Then do worst-case analysis under those assumptions

All of these can be combined with any of O, Ω, and θ

How to do case analysis

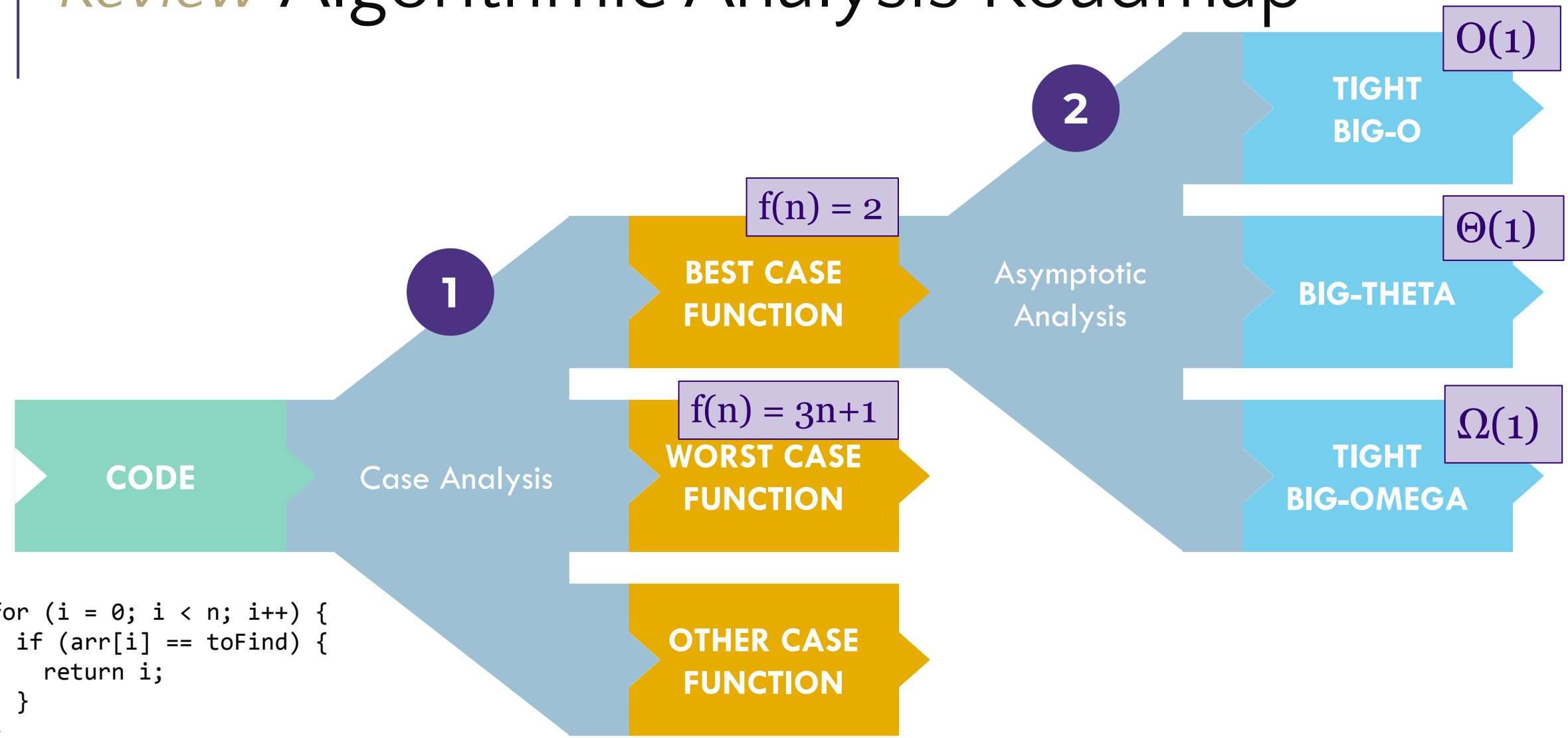
1. Look at the code, understand how thing could change depending on the state of input
   - How can you exit loops early?
   - Can you return (exit the method) early?
   - Are some if/else branches much slower than others?
2. Figure out what input **values** can cause you to hit the (best/worst) parts of the code.
   - not to be confused with number of inputs
3. Now do the analysis like normal!

Algorithmic Analysis Roadmap




```
for (i = 0; i < n; i++) {
  if (arr[i] == toFind) {
    return i;
  }
}
return -1;
```

Review Algorithmic Analysis Roadmap

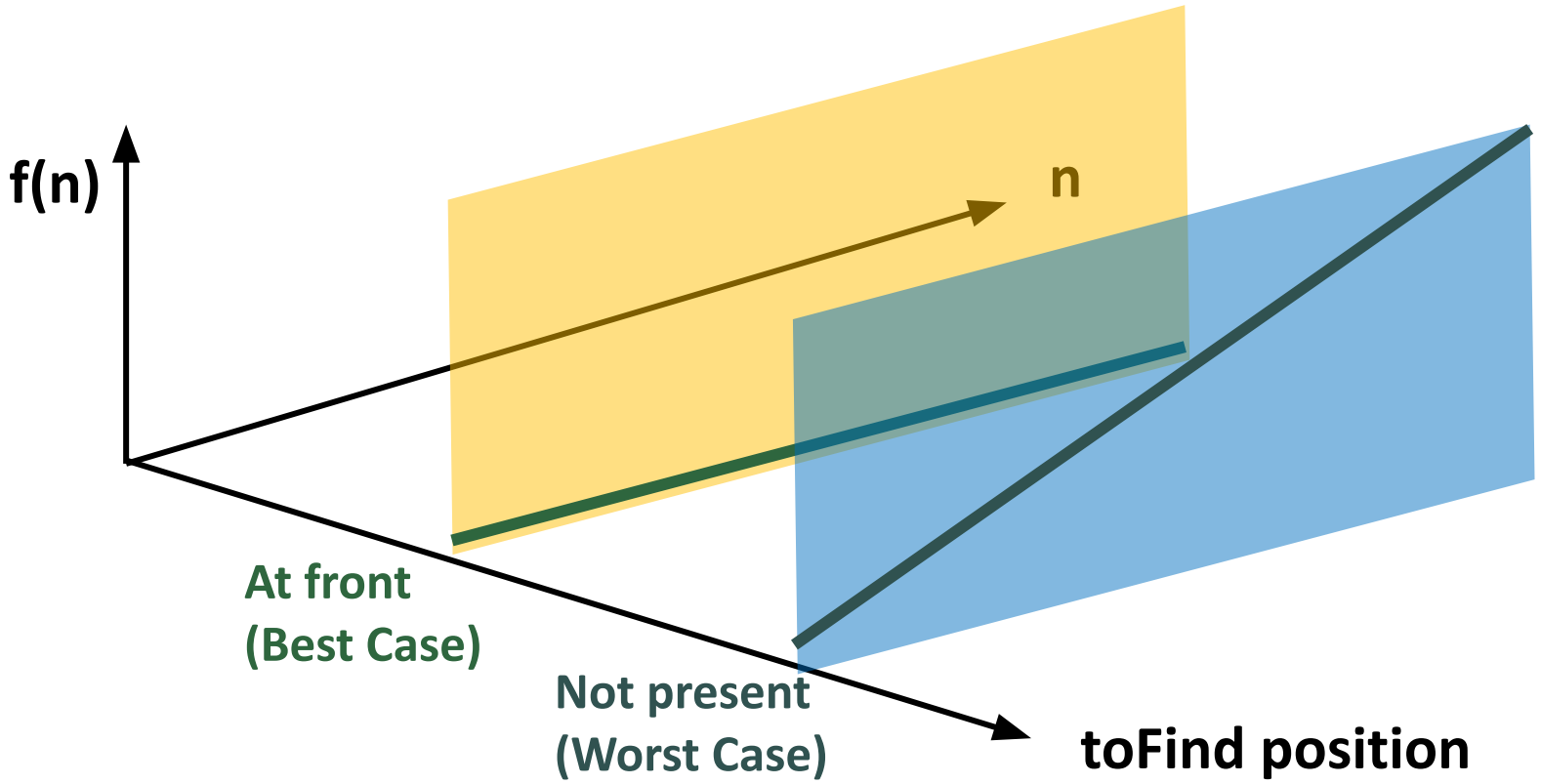


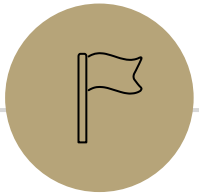

```
for (i = 0; i < n; i++) {
  if (arr[i] == toFind) {
    return i;
  }
}
return -1;
```

When to do Case Analysis?

Imagine a 3-dimensional plot

- Which case we're considering is one dimension
- Choosing a case lets us take a "slice" of the other dimensions: n and f(n)
- We do asymptotic analysis on each slice in step 2





Case Analysis

Modeling Recursive Code

Summations

Recursive Patterns

Modeling and analyzing recursive code is all about finding patterns in how the input changes between calls and how much work is done within each call

Let's explore some of the more common recursive patterns

- **Pattern #1:** Halving the Input
- **Pattern #2:** Constant size input and doing work
- **Pattern #3:** Doubling the Input

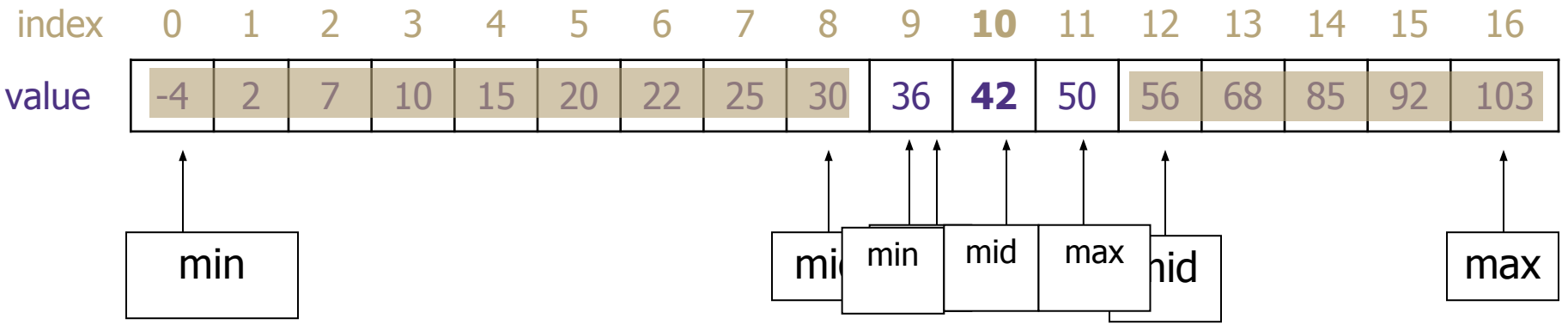
Binary Search

```
public int binarySearch(int[] arr, int toFind, int lo, int hi) {
     if( hi < lo ) {
         return -1;
     } if(hi == lo) {
         if(arr[hi] == toFind) {
             return hi;
         }
         return -1;
     }
     int mid = (lo+hi) / 2;
     if(arr[mid] == toFind) {
         return mid;
     } else if(arr[mid] < toFind) {
         return binarySearch(arr, toFind, mid+1, hi);
     } else {
         return binarySearch(arr, toFind, lo, mid-1);
     }
 }
```

Binary Search Runtime

binary search: Locates a target value in a *sorted* array or list by successively eliminating half of the array from consideration.

- Example: Searching the array below for the value **42**:



How many elements will be examined?

- What is the best case?

  element found at index 8, 1 item examined, O(1)

- What is the worst case?

  element not found, ½ elements examined, then ½ of that…

Take a guess! What is the tight Big-O of worst case binary search?

Pattern #1 – Halving the input

Binary search runtime

For an array of size N, it eliminates ½ until 1 element remains.

N, N/2, N/4, N/8, …, 4, 2, 1

- How many divisions does it take?

Think of it from the other direction:

- How many times do I have to multiply by 2 to reach N?

  1, 2, 4, 8, …, N/4, N/2, N
- Call this number of multiplications "x".

$2^x = N$

$\mathbf{x = \log_2 N}$

Binary search is in the **logarithmic** complexity class.

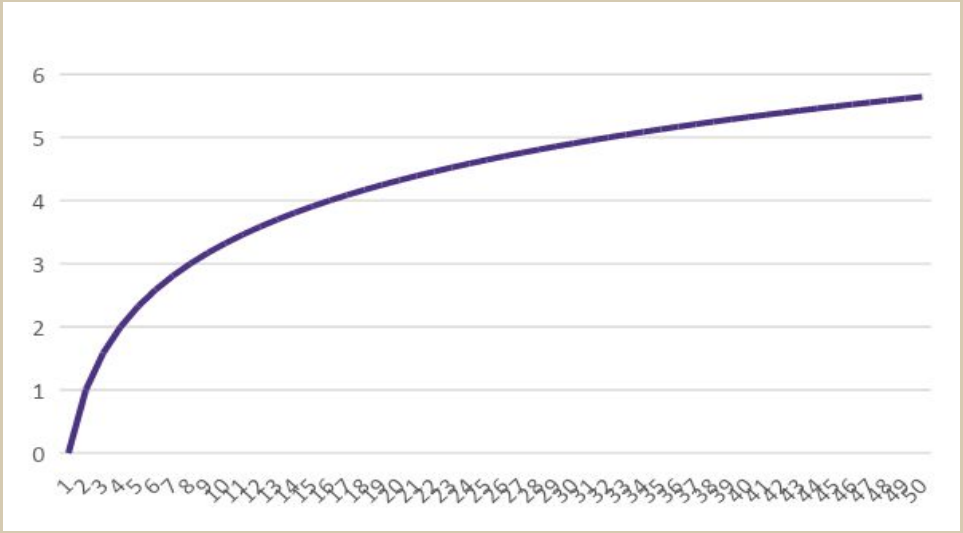
Logarithm – inverse of exponentials

$y = \log_b x \ \textit{is equal to} \ b^y = x$

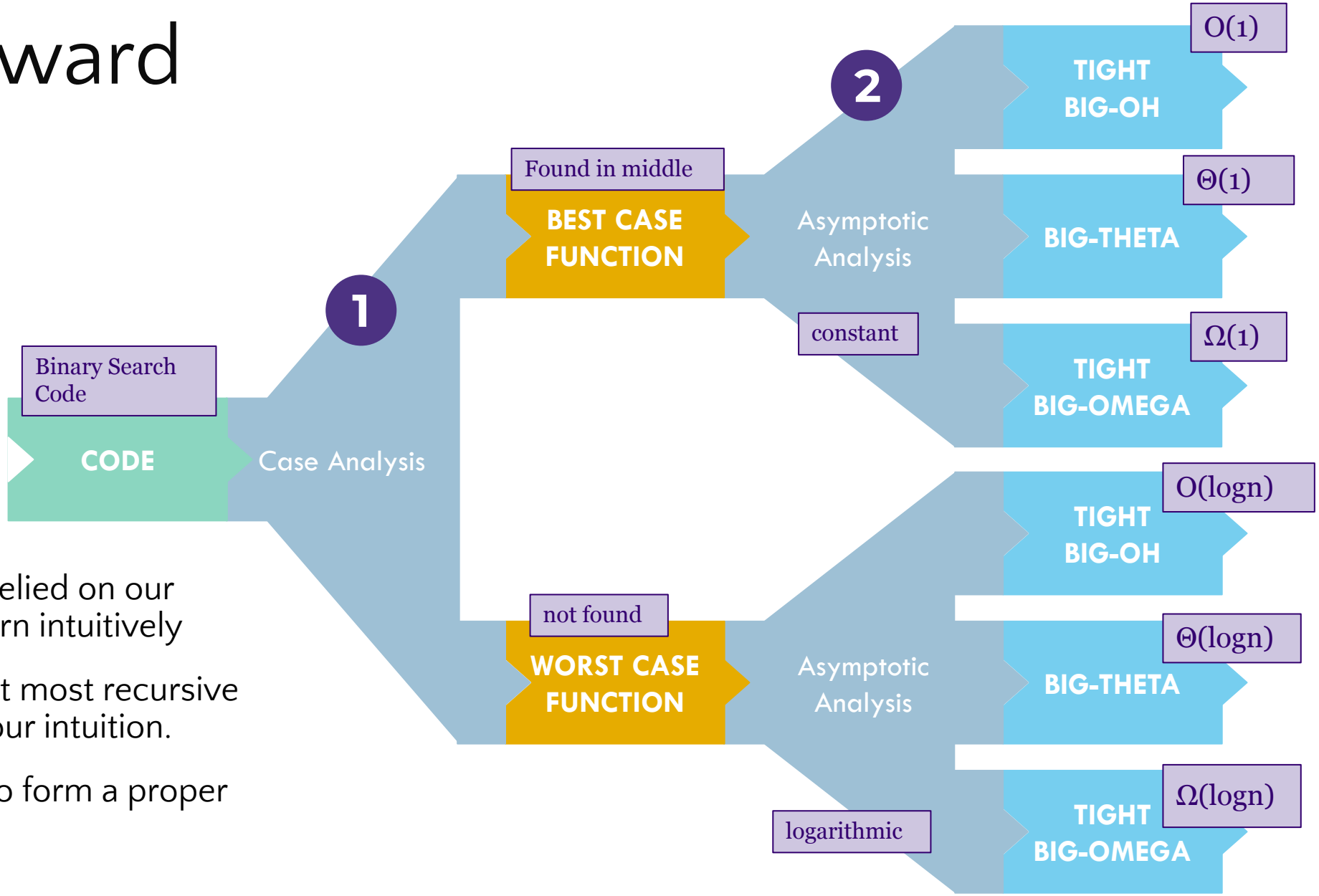
Examples:

$2^2 = 4 \Rightarrow 2 = \log_2 4$

$3^2 = 9 \Rightarrow 2 = \log_3 9$



Moving Forward



While this analysis is correct it relied on our ability to think through the pattern intuitively

This works for binary search, but most recursive code is too complex to rely on our intuition.

We need more powerful tools to form a proper code model.

Model

Let's start by just getting a model. Let $F(n)$ be our model for the worst-case running time of binary search.

```
public int binarySearch(int[] arr, int toFind, int lo, int hi) {
    if( hi < lo ) {                                          +1
        return -1;                                           +1
} else if(hi == lo) {                                        +1
        if(arr[hi] == toFind) {                              +2
            return hi;                                       +1
        }
        return -1;                                           +1
    }
    int mid = (lo+hi) / 2;                                   +2
    if(arr[mid] == toFind) {                                 +2
        return mid;                                          +1
    } else if(arr[mid] < toFind) {                           +2
        return binarySearch(arr, toFind, mid+1, hi);
    } else {
        return binarySearch(arr, toFind, lo, mid-1);
    }
}
```

worst case
+6

worst case
+6 + recursion??

How do you model recursive calls?

With a recursive math function!

Meet the Recurrence

A **recurrence** relation is an equation that defines a sequence based on a rule that gives the next term as a function of the previous term(s)

It's a lot like recursive code:

- At least one base case and at least one recursive case
- Each case should include the values for n to which it corresponds
- The recursive case should reduce the input size in a way that eventually triggers the base case
- The cases of your recurrence usually correspond exactly to the cases of the code

$$T(n) = \begin{cases} 5 & \text{if } n < 3 \\ 2T\left(\frac{n}{2}\right) + 10 & \text{otherwise} \end{cases}$$

Write a Recurrence

```
public int recursiveFunction(int n){
    if(n < 3) {
        return 3;
    }
    for(int int i=0; i < n; i++) {
        System.out.println(i);
    }
    int val1 = recursiveFunction(n/3);
    int val2 = recursiveFunction(n/3);
    return val1 * val2;
}
```

+1

+1

base case: +2

+1

*n

non-recursive work: n+2

recursive work: 2T(n/3)

+2

$$T(n) = \begin{cases} 2 & \text{if } n < 3 \\ 2T\left(\frac{n}{3}\right) + n & \text{otherwise} \end{cases}$$

Recurrence to Big-Θ

$$T(n) = \begin{cases} 2 & \text{if } n < 3 \\ 2T\left(\frac{n}{3}\right) + n & \text{otherwise} \end{cases}$$

It's still really hard to tell what the big-O is just by looking at it.

But fancy mathematicians have a formula for us to use!

Master Theorem

$$T(n) = \begin{cases} d & \text{if } n \text{ is at most some constant} \\ aT\left(\frac{n}{b}\right) + f(n) & \text{otherwise} \end{cases}$$

Where $f(n)$ is $\Theta(n^c)$

If $\log_b a < c$ then $T(n) \in \Theta(n^c)$

If $\log_b a = c$ then $T(n) \in \Theta(n^c \log n)$

If $\log_b a > c$ then $T(n) \in \Theta\left(n^{\log_b a}\right)$

$a=2\ b=3$ *and* $c=1$

$y = \log_b x$ *is equal to* $b^y = x$

$\log_3 2 = x \Rightarrow 3^x = 2 \Rightarrow x \cong 0.63$

$\log_3 2 < 1$

We're in case 1

$T(n) \in \Theta(n)$

Understanding Master Theorem

Master Theorem

$$T(n) = \begin{cases} d & \text{if } n \text{ is at most some constant} \\ aT\left(\frac{n}{b}\right) + f(n) & \text{otherwise} \end{cases}$$

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- A measures how many recursive calls are triggered by each method instance
- B measures the rate of change for input
- C measures the dominating term of the non recursive work within the recursive method
- D measures the work done in the base case

The log of a < c case

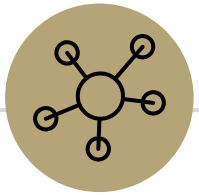
- Recursive case does a lot of non recursive work in comparison to how quickly it divides the input size
- Most work happens in beginning of call stack
- Non recursive work in recursive case dominates growth, n^c term

The log of a = c

- Recursive case evenly splits work between non recursive work and passing along inputs to subsequent recursive calls
- Work is distributed across call stack

The log of a > c case

- Recursive case breaks inputs apart quickly and doesn't do much non recursive work
- Most work happens near bottom of call stack



Questions?

Recursive Patterns

- **Pattern #1:** Halving the Input

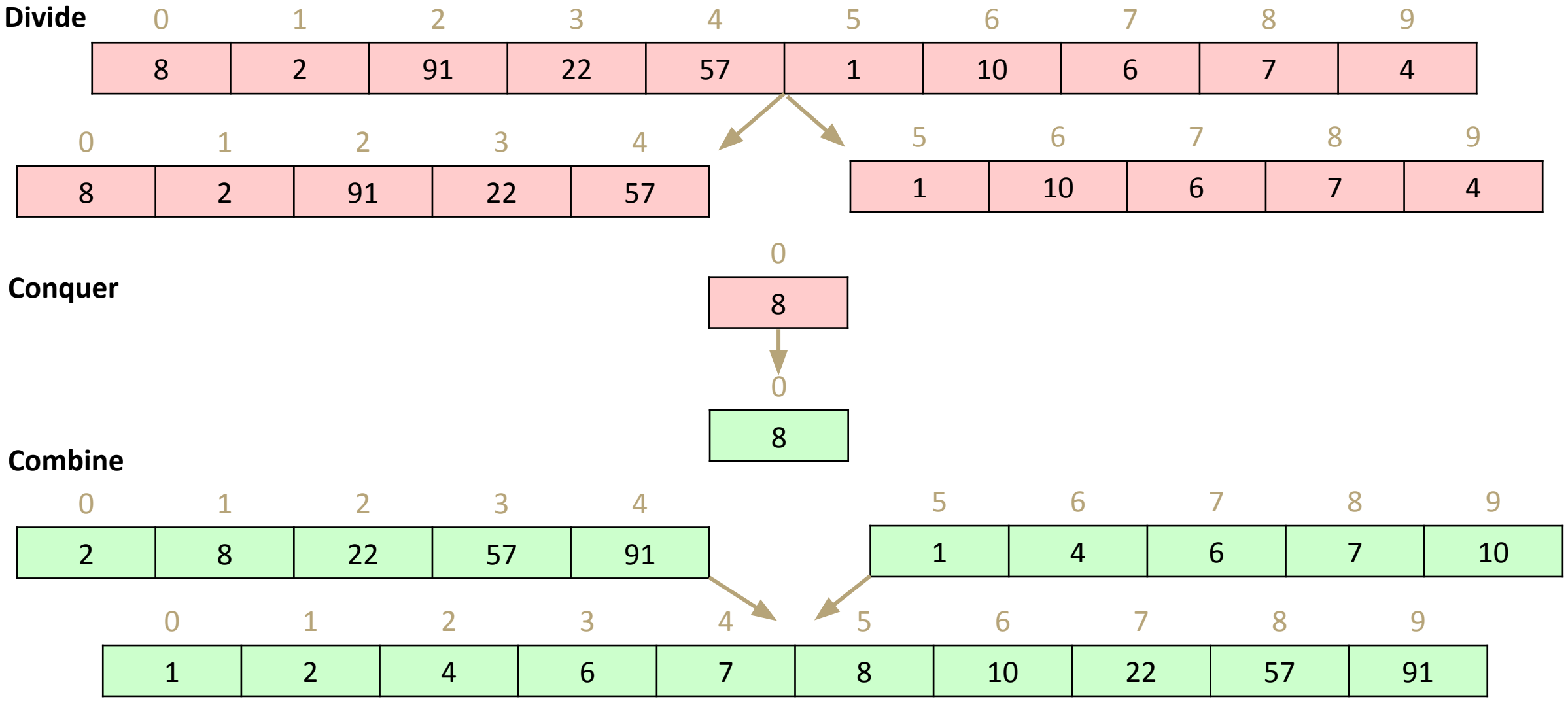
  Binary Search $\Theta(\log n)$

- **Pattern #2:** Constant size input and doing work

  Merge Sort

- **Pattern #3:** Doubling the Input

Merge Sort


Divide
0 1 2 3 4 5 6 7 8 9
8 2 91 22 57 1 10 6 7 4
0 1 2 3 4
8 2 91 22 57
5 6 7 8 9
1 10 6 7 4
Conquer
0
8
0
8
Combine
0 1 2 3 4
2 8 22 57 91
5 6 7 8 9
1 4 6 7 10
0 1 2 3 4 5 6 7 8 9
1 2 4 6 7 8 10 22 57 91

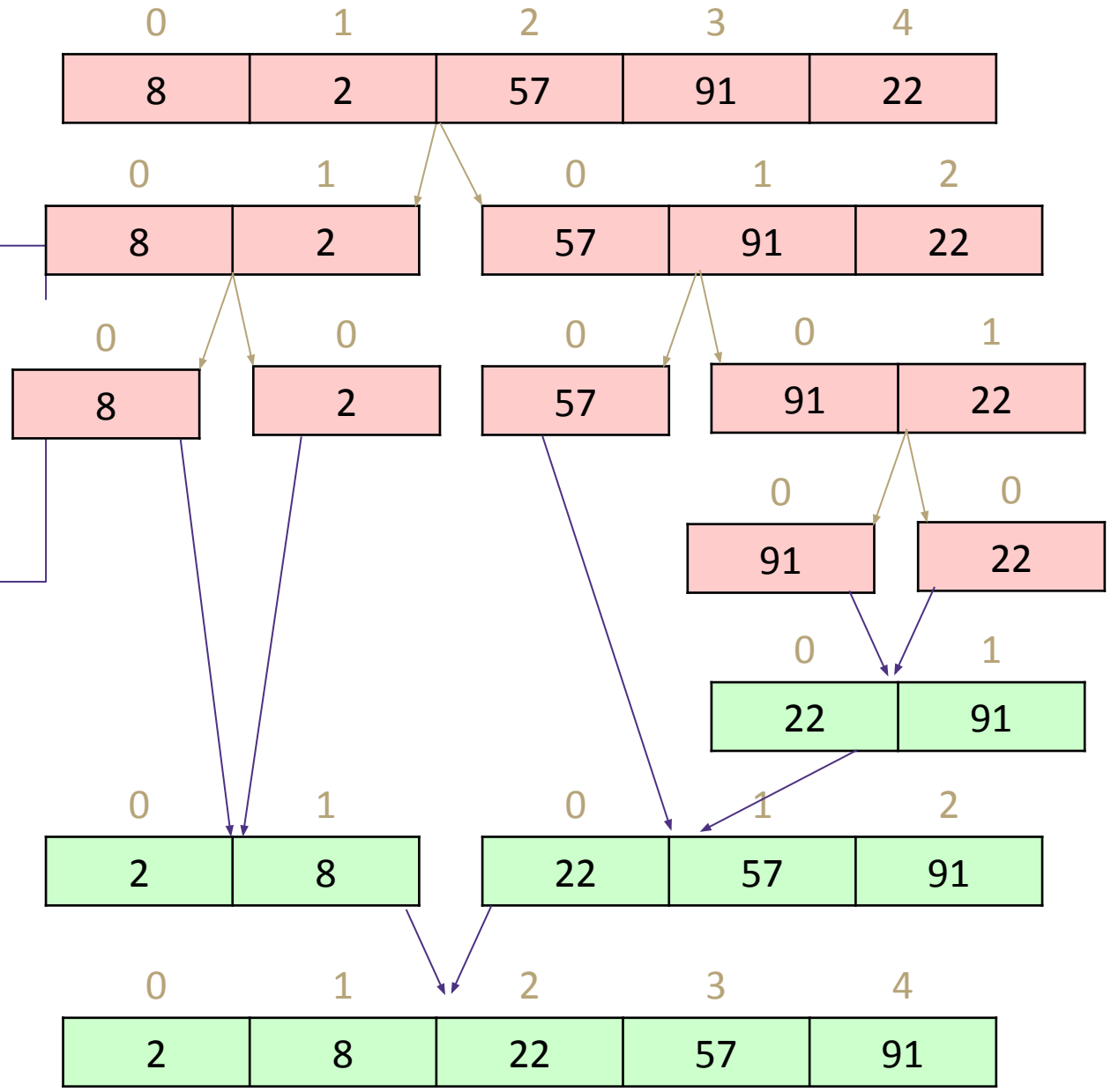
Merge Sort

```
mergeSort(input) {
    if (input.length == 1)
        return
    else
        smallerHalf = mergeSort(new [0, ..., mid])
        largerHalf = mergeSort(new [mid + 1, ...])
        return merge(smallerHalf, largerHalf)
}
```

$$T(n) = \begin{cases} 1 \text{ if } n <= 1 \\ 2T(n/2) + n \text{ otherwise} \end{cases}$$

Pattern #2 – Constant size input and doing work

Take a guess! What is the Big-O of worst case merge sort?



Merge Sort Recurrence to Big-Θ

$$T(n) = \begin{cases} 1 \text{ if } n <= 1 \\ 2T(n/2) + n \text{ otherwise} \end{cases}$$

Master Theorem

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Where $f(n)$ is $\Theta(n^c)$

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If $\log_b a > c$ then $T(n) \in \Theta\left(n^{\log_b a}\right)$

a=2 b=2 and c=1

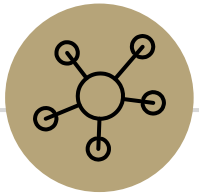
$y = \log_b x$ *is equal to* $b^y = x$

$\log_2 2 = x \Rightarrow 2^x = 2 \Rightarrow x = 1$

$\log_2 2 = 1$

We're in case 2

$T(n) \in \Theta(n \log n)$



Questions?

Recursive Patterns

- **Pattern #1:** Halving the Input

  Binary Search Θ(logn)

- **Pattern #2:** Constant size input and doing work

  Merge Sort Θ(nlogn)

- **Pattern #3:** Doubling the Input

  Calculating Fibonacci

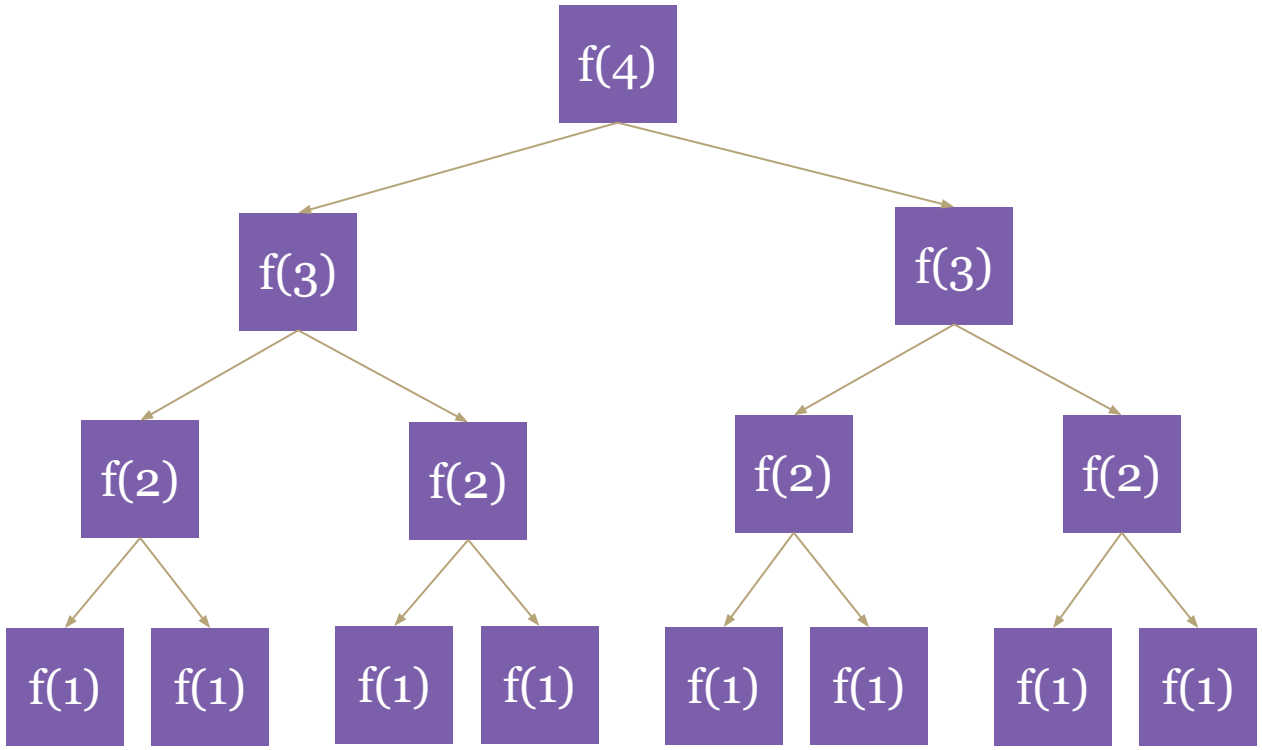
Calculating Fibonacci

```
public int fib(int n) {
    if (n <= 1) {
        return 1;
    }
    return fib(n-1) + fib(n-1);
}
```

- Each call creates 2 more calls
- Each new call has a copy of the input, almost
- Almost doubling the input at each call

Almost

Pattern #3 – Doubling the Input



Calculating Fibonacci Recurrence to Big-Θ

```
public int f(int n) {
    if (n <= 1) {
        return 1;
    }
    return f(n-1) + f(n-1);
}
```

d

2T(n-1) + c

$$T(n) = \begin{cases} d \text{ when } n \leq 1 \\ 2T(n-1) + c \text{ otherwise} \end{cases}$$

Can we use master theorem?

Master Theorem

$$T(n) = \begin{cases} d & \text{if } n \text{ is at most some constant} \\ aT\left(\frac{n}{b}\right) + f(n) & \text{otherwise} \end{cases}$$

Uh oh, our model doesn't match that format...

Can we intuit a pattern? ("**unrolling**")

T(1) = d
T(2) = 2T(2-1) + c = 2(d) + c
T(3) = 2T(3-1) + c = 2(2(d) + c) + c = 4d + 3c
T(4) = 2T(4-1) + c = 2(4d + 3c) + c = 8d + 7c
T(5) = 2T(5-1) + c = 2(8d + 7c) + c = 16d +25c

Looks like something's happening but it's tough
Maybe geometry can help!

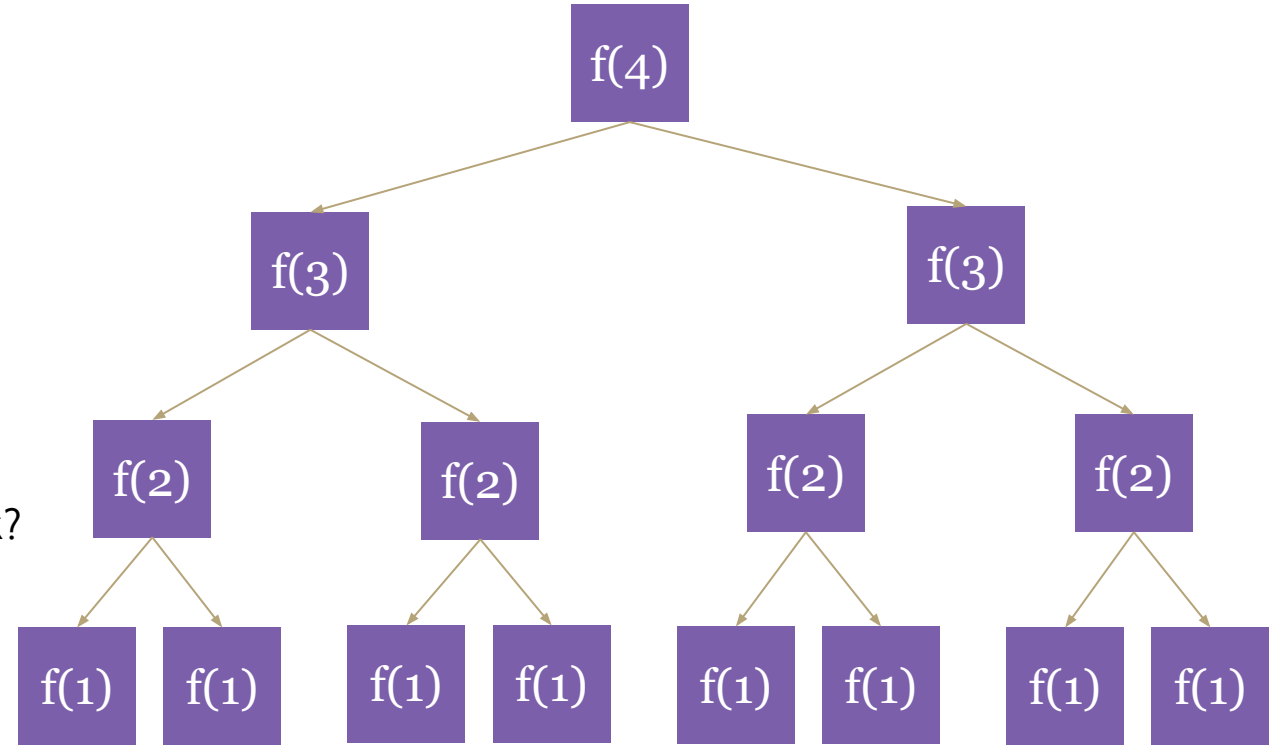
Calculating Fibonacci Recurrence to Big-Θ

$$T(n) = \begin{cases} d \text{ when } n \leq 1 \\ 2T(n-1) + c \text{ otherwise} \end{cases}$$

How many layers in the function call tree?

How many layers will it take to transform "n" to the base case of "1" by subtracting 1

For our example, 4 -> Height = n



How many function calls per layer?

Layer	Function calls
1	1
2	2
3	4
4	8

How many function calls on layer k?

2^{k-1}

How many function calls TOTAL for a tree of k layers?

$1 + 2 + 3 + 4 + \ldots + 2^{k-1}$

Calculating Fibonacci Recurrence to Big-Θ

Patterns found:

How many layers in the function call tree? n

How many function calls on layer k? 2^{k-1}

How many function calls TOTAL for a tree of k layers?

$$1 + 2 + 4 + 8 + \ldots + 2^{k-1}$$

Total runtime = (total function calls) x (runtime of each function call)

Total runtime = $(1 + 2 + 4 + 8 + \ldots + 2^{k-1})$ x (constant work)

$$1 + 2 + 4 + 8 + \ldots + 2^{k-1} = \sum_{i=1}^{k-1} 2^i = \frac{2^k - 1}{2 - 1} = 2^k - 1$$

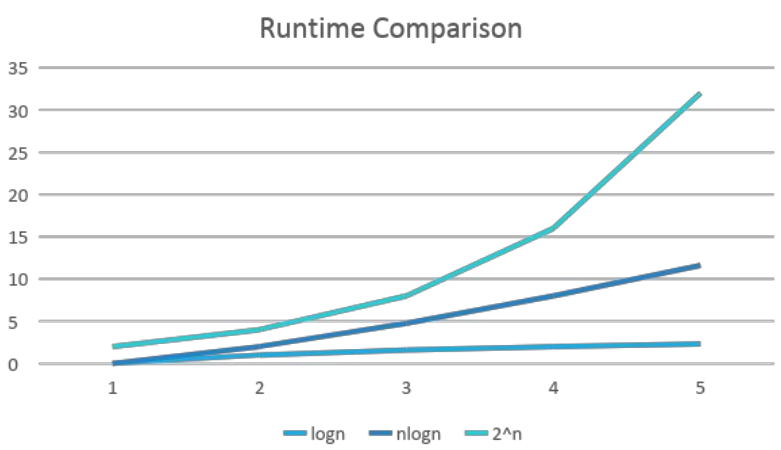
$$\boldsymbol{T(n) = 2^n - 1 \in \Theta(2^n)}$$

Summation Identity
Finite Geometric Series

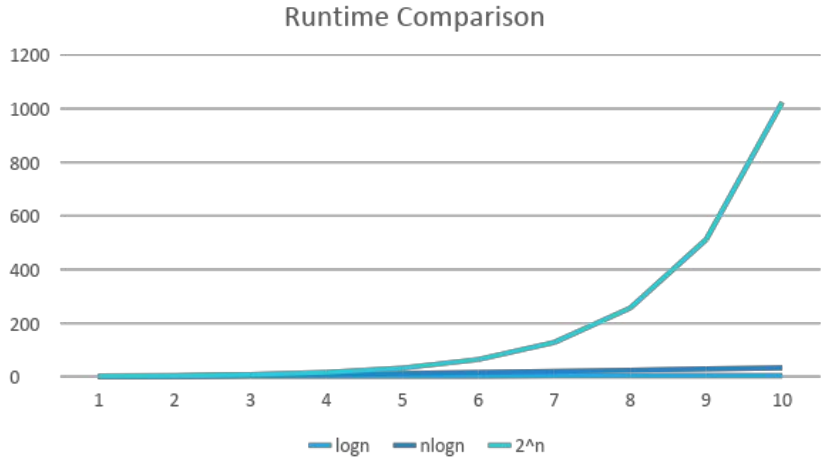
$$\sum_{i=1}^{k-1} x^i = \frac{x^k - 1}{x - 1}$$

Recursive Patterns

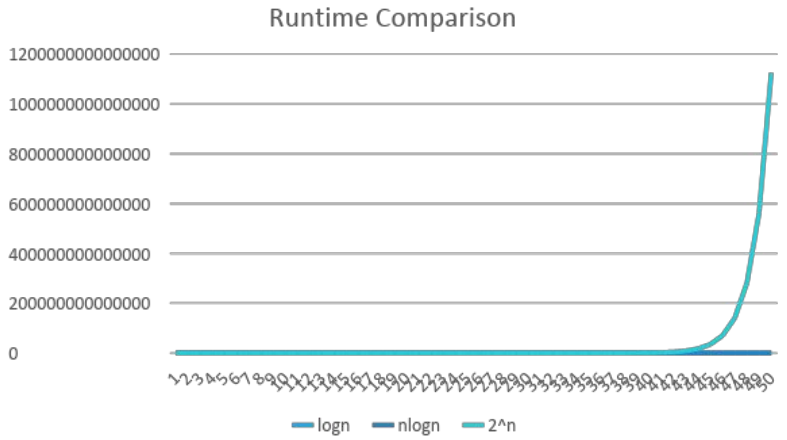
Pattern #1: Halving the Input
Binary Search Θ(logn)

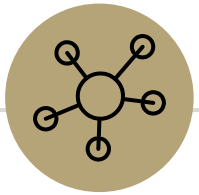


Pattern #2: Constant size input and doing work
Merge Sort Θ(nlogn)

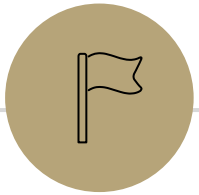


Pattern #3: Doubling the Input
Calculating Fibonacci Θ(2^n)

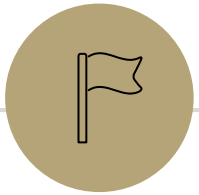




Questions?



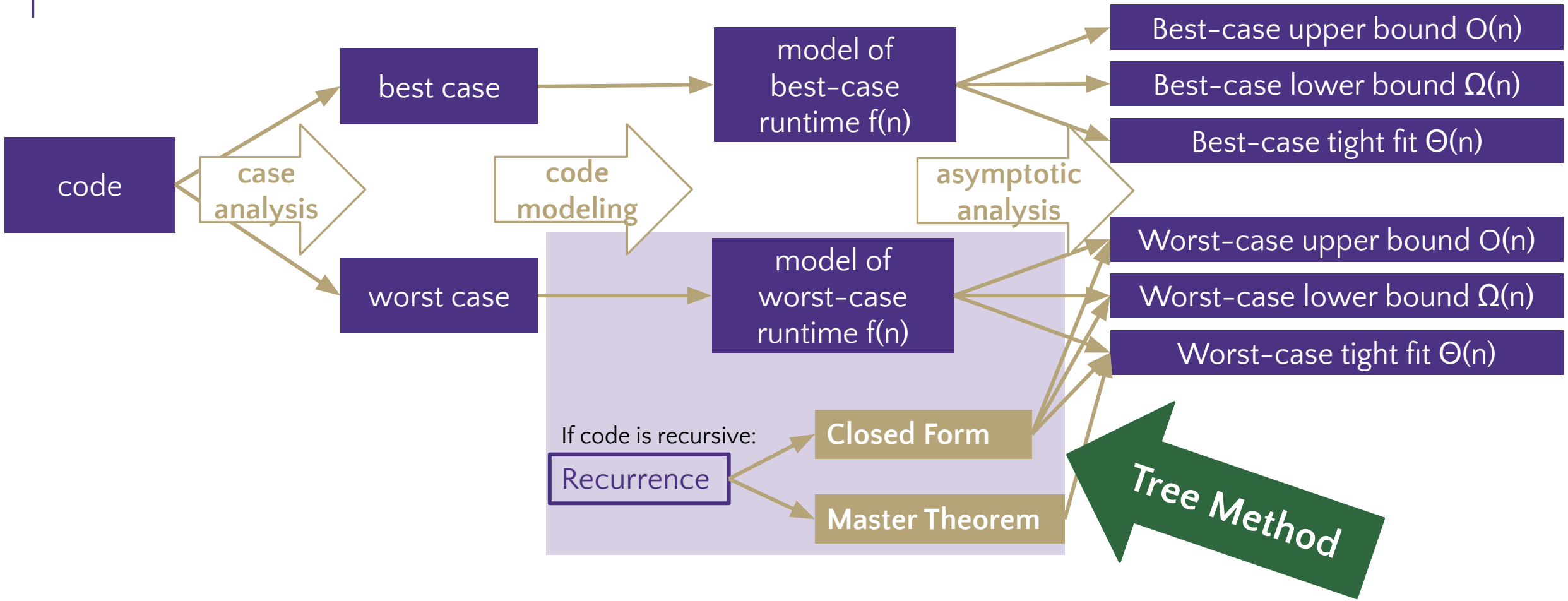
That’s all!



Appendix

Extra slides

Code Analysis Process



Recurrence to Big Θ Techniques

A recurrence is a mathematical function that includes itself in its definition

This makes it very difficult to find the dominating term that will dictate the asymptotic growth

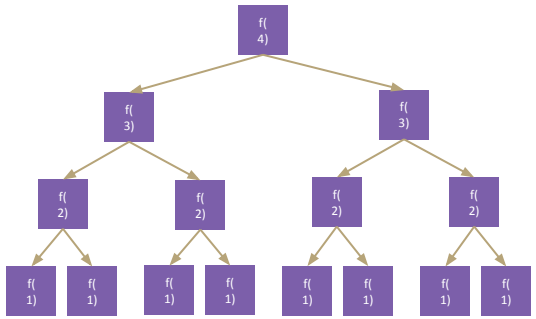
Solving the recurrence or "finding the closed form" is the process of eliminating the recursive definition. So far, we've seen three methods to do so:

1. Apply Master Theorem
   - Pro: Plug and chug convenience
   - Con: only works for recurrences of a certain format
2. Unrolling
   - Pro: Least complicated setup
   - Con: requires intuitive pattern matching
3. Tree Method
   - Pro: Plug and chug
   - Con: Complex setup

Master Theorem

$$T(n) = \begin{cases} d \text{ when } n \leq 1 \\ 2T(n-1) + c \text{ otherwise} \end{cases}$$

$T(1) = d$
$T(2) = 2T(2-1) + c = 2(d) + c$
$T(3) = 2T(3-1) + c = 2(2(d) + c) + c = 4d + 3c$
$T(4) = 2T(4-1) + c = 2(4d + 3c) + c = 8d + 7c$
$T(5) = 2T(5-1) + c = 2(8d + 7c) + c = 16d + 25c$



Tree Method

Draw out call stack, what is the input to each call? How much work is done by each call?

How much work is done at each layer?

- 64 for this example -> n work at each layer
- Work is variable per layer, but across the entire layer work is constant - always n

How many layers are in our function call tree?

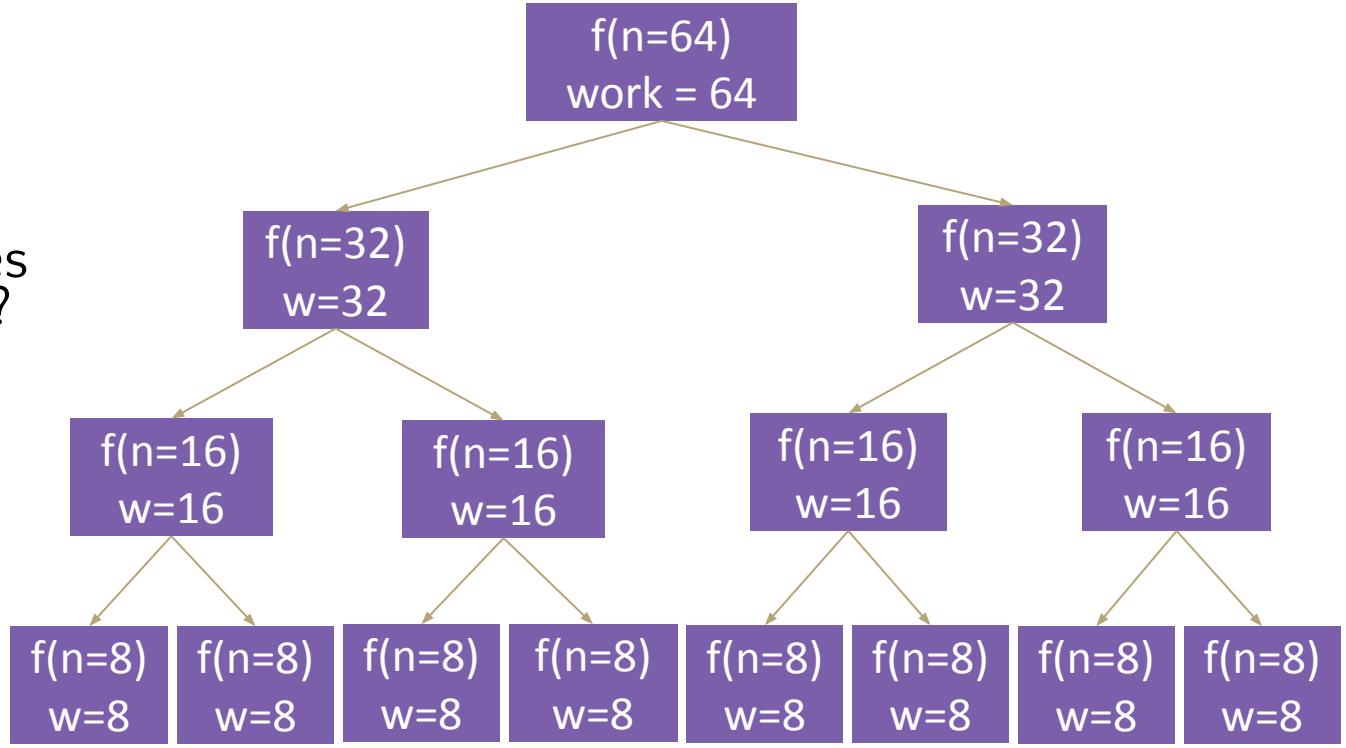
Hint: how many levels of recursive calls does it take *binary search* to get to the base case?

Height = $\log_2 n$

It takes $\log_2 n$ divisions by 2 for n to be reduced to the base case 1

$\log_2 64 = 6$ -> 6 levels of this tree

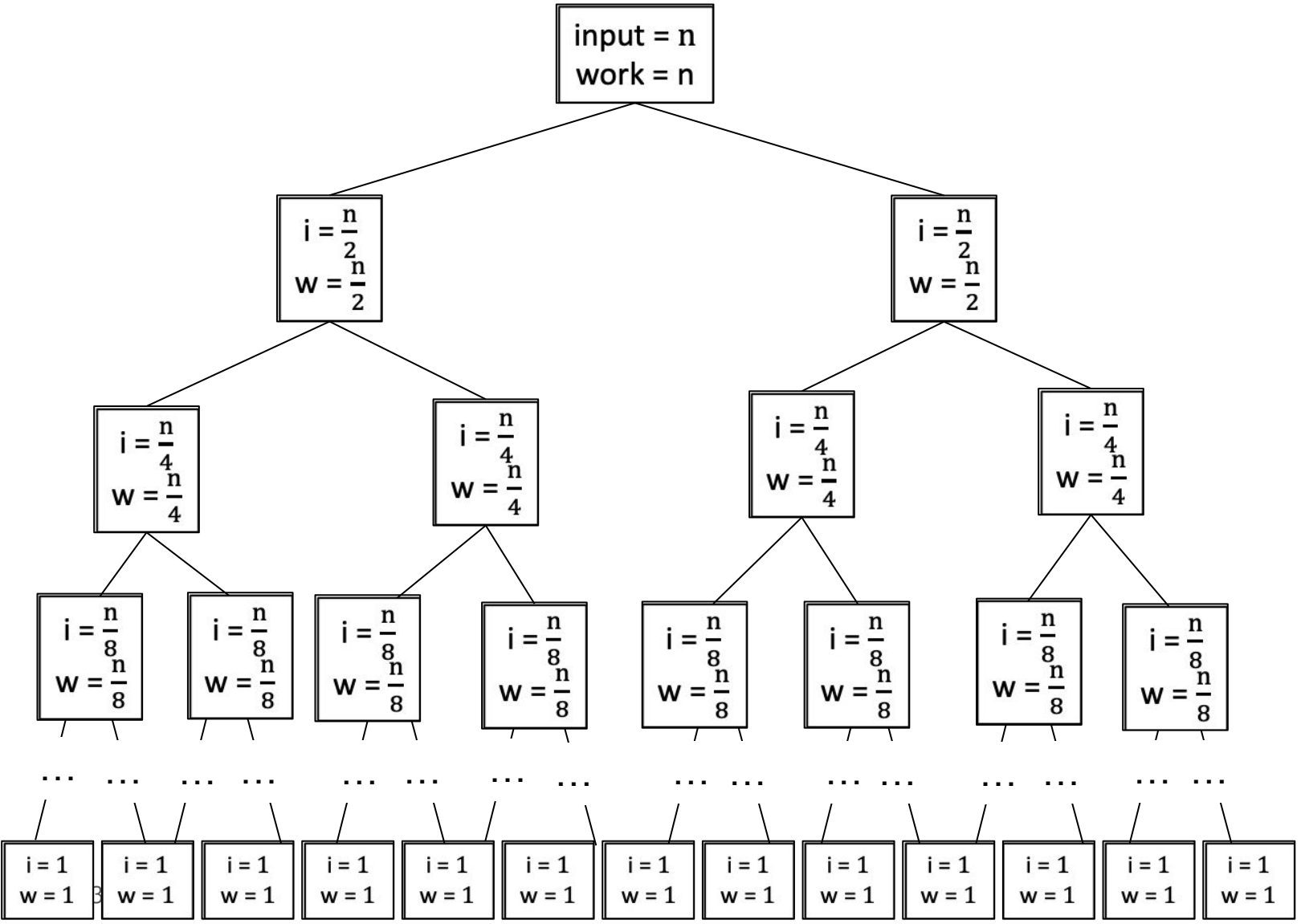
Merge Sort $T(n) = \begin{cases} 1 \text{ if } n <= 1 \\ 2T(n/2) + n \text{ otherwise} \end{cases}$



… and so on…

Tree Method

$$T(n) = \begin{cases} 1 \text{ when } n \leq 1 \\ 2T\left(\frac{n}{2}\right) + n \text{ otherwise} \end{cases}$$



Recursive level	How many nodes at each level?	How much work done by each node?	How much work across each level?
0	1	n	n
1	2	$\frac{n}{2}$	n
2	4	$\frac{n}{4}$	n
3	8	$\frac{n}{8}$	n
logn	n	1	n

Tree Method Practice

$$T(n) = \begin{cases} 1 \text{ when } n \leq 1 \\ 2T\left(\frac{n}{2}\right) + n \text{ otherwise} \end{cases}$$

1. What is the size of the input on level i? $\frac{n}{2^i}$
2. What is the work done by each node on the i^{th} recursive level? $(\frac{n}{2^i})$
3. What is the number of nodes at level i? 2^i
4. What is the total work done at the i^{th} recursive level?

   $numNodes * workPerNode = 2^i\left(\frac{n}{2^i}\right) = n$
5. What value of i does the last level occur?

   $\frac{n}{2^i} = 1 \rightarrow n = 2^i \rightarrow i = \log_2 n$
6. What is the total work across the base case level?

   $numNodes * workPerNode = 2^{log_2 n}(1) = n$

Level (i)	Number of Nodes	Work per Node	Work per Level
0	1	n	n
1	2	n/2	n
2	4	n/4	n
3	8	n/8	n
$\log_2$n	n	1	

Combining it all together…

$$T(n) = \sum_{i=0}^{\log_2 n - 1} n \; + n \; = nlog_2 n + n \; = \Theta(\text{nlogn})$$

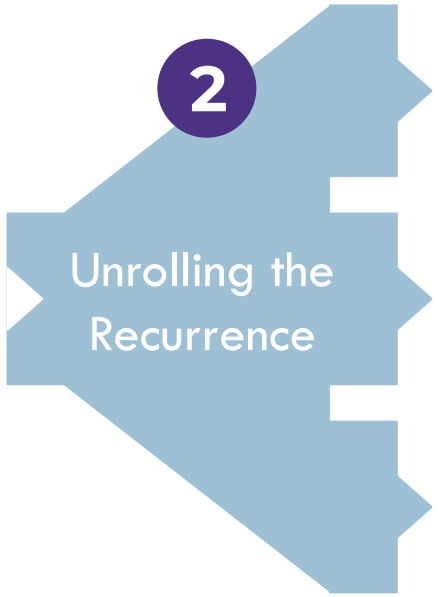
power of a log

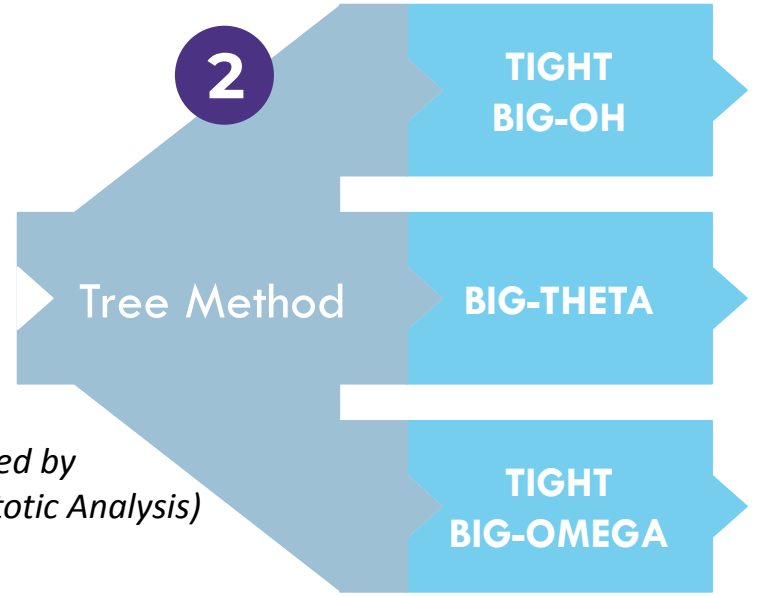
$$x^{\log_b y} = y^{\log_b x}$$

Summation of a constant

$$\sum_{i=0}^{n-1} c = cn$$

Recurrence to Big-Theta: Our Toolbox



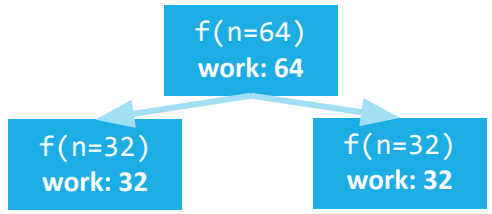


(followed by Asymptotic Analysis)

MASTER THEOREM

$$T(n) = \begin{cases} d & \text{if } n \text{ is at most some constant} \\ aT\left(\frac{n}{b}\right) + f(n) & \text{otherwise} \end{cases}$$

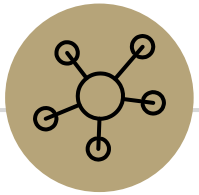
$T(1) = d$
$T(2) = 2T(2\text{-}1) + c = 2(d) + c$
$T(3) = 2T(3\text{-}1) + c = 2(2(d) + c) + c = 4d + 3c$



PROS: Convenient to plug 'n' chug
CONS: Only works for certain format of recurrences

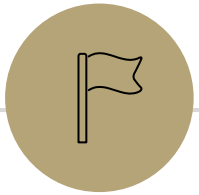
PROS: Least complicated setup
CONS: Requires intuitive pattern matching, no formal technique

PROS: Convenient to plug 'n' chug
CONS: Complicated to set up for a given recurrence



Questions?

Case Analysis
Modeling Recursive Code

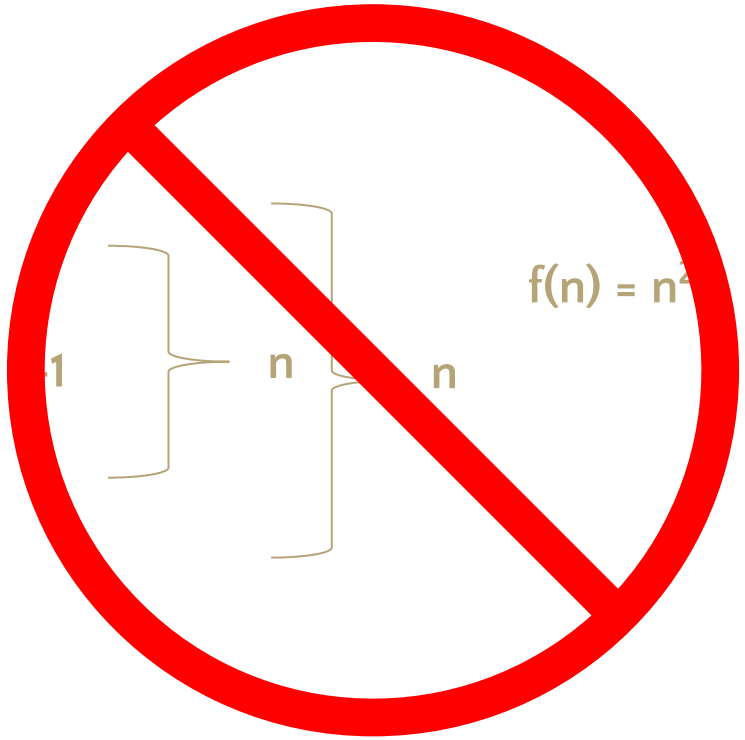


Summations

Modeling Complex Loops

Write a mathematical model of the following code

```
for (int i = 0; i < n; i++) {
    for (int j = 0; j < i; j++) {
        System.out.println("Hello!");
    }
}
```



Keep an eye on loop bounds!

Modeling Complex Loops

```
for (int i = 0; i < n; i++) {
    for (int j = 0; j < i; j++) {
        System.out.print("Hello! ");
    }
      System.out.println();
}
```

+1 ; 0 + 1 + 2 + 3 +…+ i–1 ; n

T(n) = (0 + 1 + 2 + 3 +…+ i–1)

How do we model this part?

Summations!
1 + 2 + 3 + 4 +… + n = $\sum_{i=1}^{n} i$

Definition: Summation

$$\sum_{i=a}^{b} f(i) = f(a) + f(a+1) + f(a+2) + \ldots + f(b-2) + f(b-1) + f(b)$$

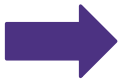
T(n) = $\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} 1$

What is the Big O?

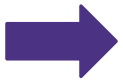
Simplifying Summations

Find closed form using summation identities (given on exams)

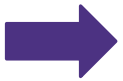
```
for (int i = 0; i < n; i++) {
   for (int j = 0; j < i; j++) {
      System.out.println("Hello!");
   }
}
```



$$T(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} 1$$



closed form



simplified tight big O

$$T(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} 1 \quad = \sum_{i=0}^{n-1} 1 \cdot i \quad = 1 \sum_{i=0}^{n-1} i \quad = \frac{n(n-1)}{2} \quad = \frac{1}{2}n^2 - \frac{1}{2}n \quad = \boldsymbol{O(n^2)}$$

Summation of a constant

$$\sum_{i=0}^{n-1} c = cn$$

Factoring out a constant

$$\sum_{i=a}^{b} cf(i) = c \sum_{i=a}^{b} f(i)$$

Gauss's Identity

$$\sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}$$