Lecture 25: P vs NP

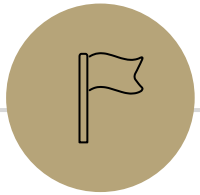
CSE 373: Data Structures and Algorithms

Announcements

Slido Event #7165864
https://app.sli.do/event/h2TaZahvBEHWg2h79kNHxC

- Final exam TA lead review after class today
- Please fill out the section final review survey to help your TAs play for tomorrow's section
- P4 – please get started on Seam Carving if you haven't already
  - TAs posted a P4 walk through to help clarify
  - Note that office hours will end on week 10
  - You cannot use late days for P4 (I already made the turn in as late as I can accept assignments)
- Please nominate your TAs for an award!
  - TO NOMINATE GO TO: https://www.cs.washington.edu/students/ta/bandes

[P vs NP Explained (YouTube)]



P vs. NP

A brief history of computer science problem solving

- The field of “computer science” is the pursuit of determining how to use “computers” to help solve human problems
- 1843 the first “computer” is designed to solve bernouli’s numbers, a very difficult calculation
- 1943 the MARC II is designed to solve missile trajectory and other military calculations
- 1960s computers begin to become more generalized, researchers start looking for ways to use computers beyond basic math computations
- 1970s researchers are exploring what types of problems computers can help with, and finding themselves stuck and unsure if there exists a computer assisted solution or not…

1970s computer science research

- Researchers were collecting problems to solve
  - Some problems resulted in algorithms that could “efficiently” find a solution
- When researchers were stuck on a problem they turned to “reductions” to see if they could apply a newly discovered algorithm to their own problem
- To help one another understand if they were working on an unsolved problem or not researchers started to categorize problems into complexity classes…
  - Enter “complexity research”

"Efficiency"

So far you've only met problems that have an "efficient" solution

For our purposes "efficient" essentially means "can be executed by current day computers"

Formally we will consider any code that can run in **polynomial** or "**P**" time to be "efficient"

P complexity class

The set of all decision problems that have an algorithm that runs in time $O(n^k)$ for some constant k

Are these algorithms always actually efficient?

Well… no

Your n^{10000} algorithm or $10000n^3$ algorithms probably aren't going to finish anytime soon, but these edge cases are rare, and polynomial time is good as a low bar

“Efficiency” at scale

We have seen some inefficient algorithms

- Recursive backtracking (k^n where k represents number of choices)
- Recursive fibonacci (2^n)

But as long as n is small we can still compute them

N^3 solution where n= 100 takes ~3 hrs

2^n solution where n = 10 takes ~milliseconds,

but n = 100 takes 300 quintillion years (longer than the age of the universe)

Running Times

TABLE 2 The Computer Time Used by Algorithms.

Problem Size	*Bit Operations Used*					
n	$\log n$	n	$n \log n$	n^2	2^n	$n!$
10	3×10^{-11} s	10^{-10} s	3×10^{-10} s	10^{-9} s	10^{-8} s	3×10^{-7} s
10^2	7×10^{-11} s	10^{-9} s	7×10^{-9} s	10^{-7} s	4×10^{11} yr	*
10^3	1.0×10^{-10} s	10^{-8} s	1×10^{-7} s	10^{-5} s	*	*
10^4	1.3×10^{-10} s	10^{-7} s	1×10^{-6} s	10^{-3} s	*	*
10^5	1.7×10^{-10} s	10^{-6} s	2×10^{-5} s	0.1 s	*	*
10^6	2×10^{-10} s	10^{-5} s	2×10^{-4} s	0.17 min	*	*

Table from Rosen's Discrete Mathematics textbook
How big of a problem can we solve for an algorithm with the given running times?
"*" means more than 10^{100} years.

Aside: Decision Problems

Today's goal is to break problems into solvable/not solvable categories

For today, we're going to talk about **decision problems.**

- Problems that have a "yes" or "no" answer.

Why?

Theory reasons / how we translate problems for computer understanding

But it's not too bad

- most problems can be rephrased as very similar decision problems

E.g. instead of "find the shortest path from s to t" ask,

- Is there a path from s to t length at most *k*?

NP Complexity Class

NP (stands for "nondeterministic polynomial")

The set of all decision problems such that if the answer is YES, there is a proof of that which can be verified in polynomial time

Decision Problems such that:

- If the answer is YES, you can prove the answer is yes by
  - A given "proof" or a "certificate" can be verified in polynomial time
  - Puzzle problems where a given answer can be either confirmed or rejected
- What certificate would be convenient for short paths?
  - The path itself. Easy to check the path is really in the graph and really short.

Light Spanning Tree:
IS there a spanning tree of graph G of weight at most k?

The spanning tree itself.
Verify by checking it really connects every vertex and its weight.

2-Coloring:
Can you color vertices of a graph red and blue so every edge has differently colored endpoints?

The coloring.
Verify by checking each edge.

2-SAT:
Given a set of variables and a list of requirements:
(variable==[T/F] || variable==[T/F])
Find a setting of the variables to make every requirement true.

The assignment of variables.
Verify by checking each requirement.

P vs. NP, the conundrum

P vs. NP

Are P and NP the same complexity class?
That is, can every problem that can be verified in polynomial time also be solved in polynomial time.

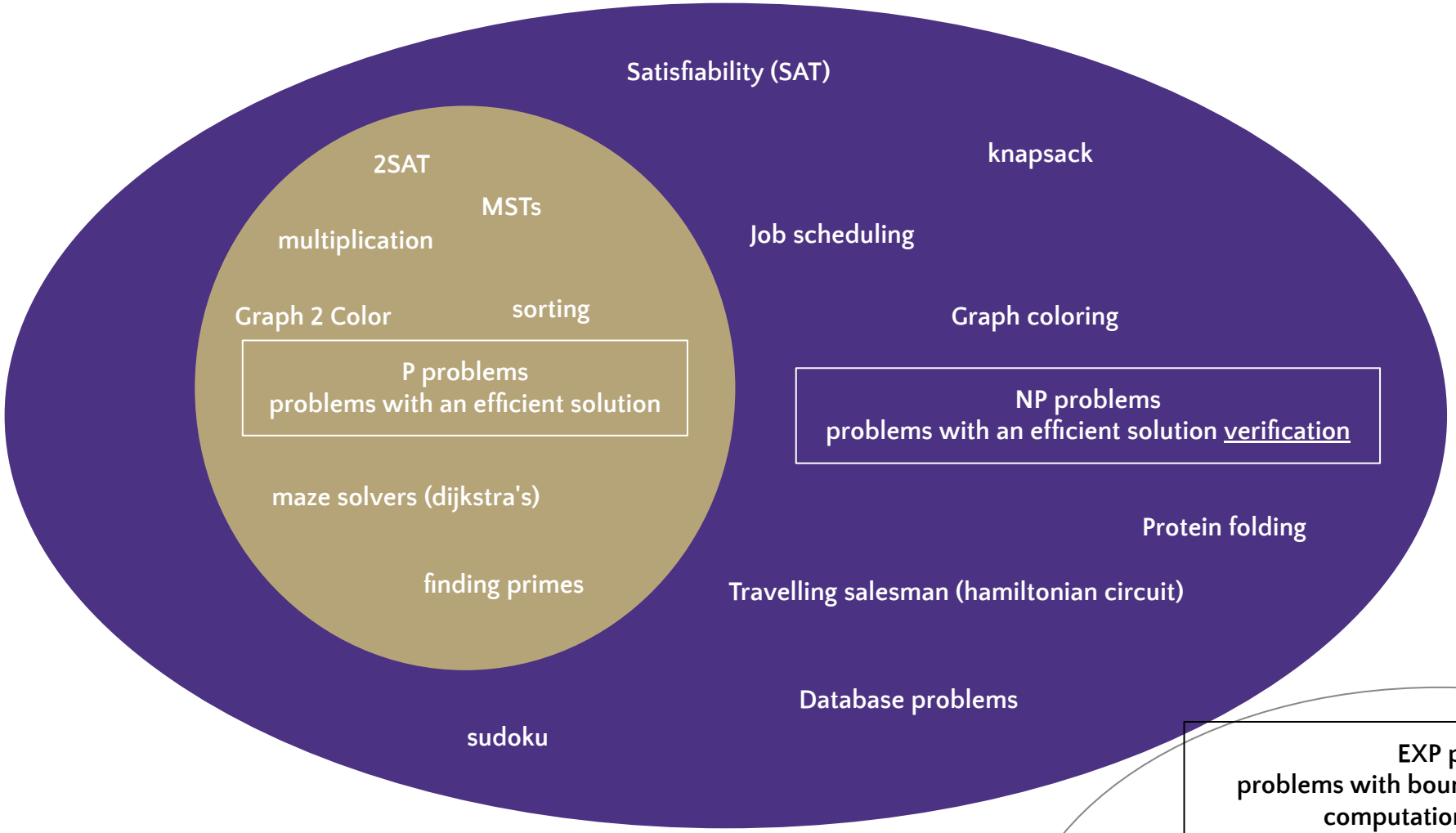
Does being able to quickly validate a correct solution also mean you can quickly find a correct solution?

No one knows the answer to this question.

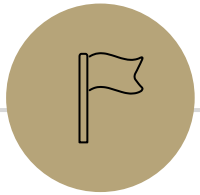
In fact, it's the biggest open problem in Computer Science.

P vs NP

Can we **PROVE** that all problems with an efficiently verifiable solution can be solved efficiently?



Will all NP problems be discovered to also be in P?



Searching for a solution to P v NP

Hard Problems

Let's say we want to prove that every problem in NP can actually be solved efficiently.

We might want to start with a really hard problem in NP.

What is the hardest problem in NP?

What does it mean to be a hard problem?

Reductions are a good definition:

- If A reduces to B then "A ≤ B" (in terms of difficulty)
  - Once you have an algorithm for B, you have one for A automatically from the reduction!

Does there exist an algorithm that all NP problems reduce to?

NP-Completeness

NP-complete

The problem B is NP-complete if B is in NP and
for all problems A in NP, A reduces to B.

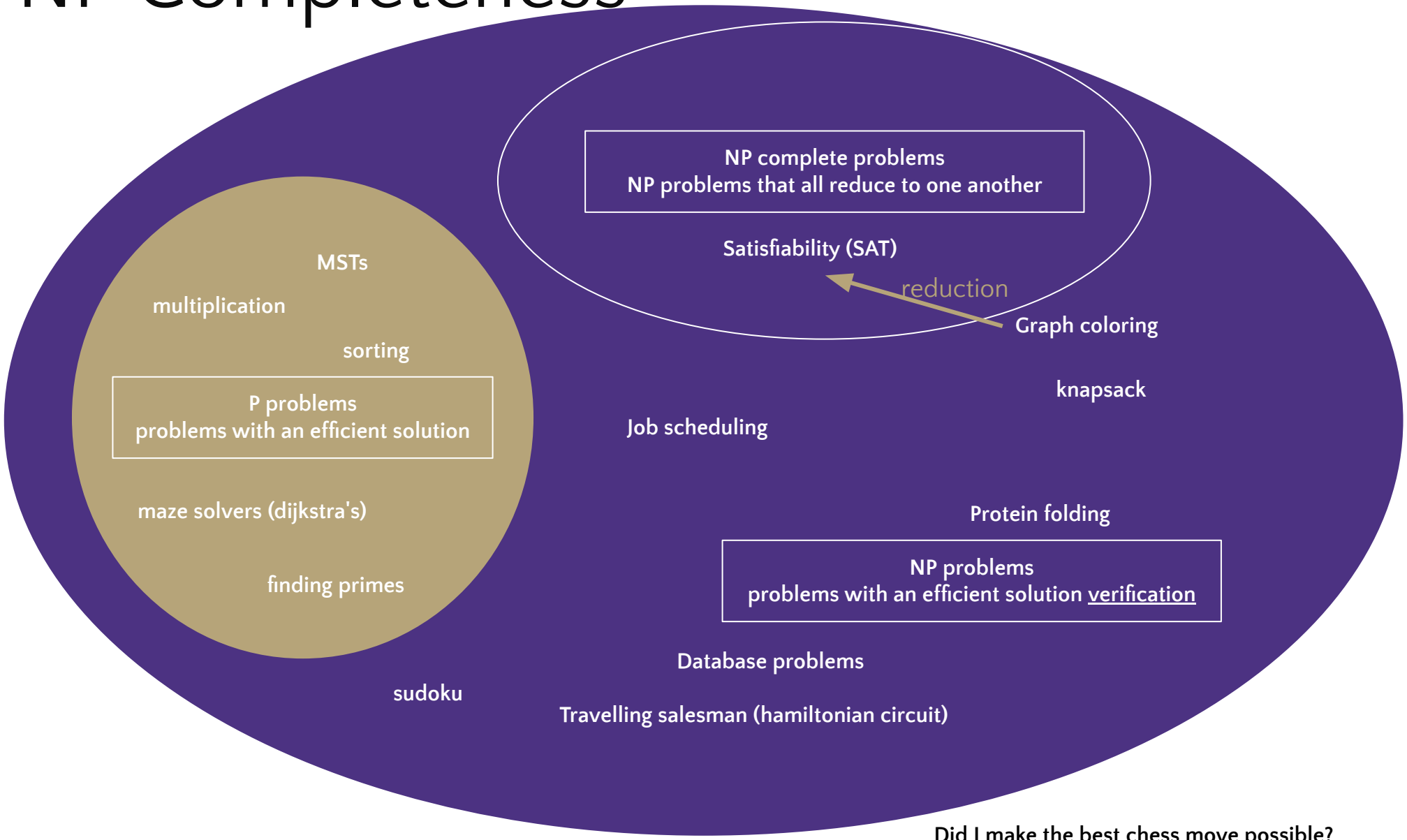
An NP-complete problem is a "hardest" problem in NP.

If you have an algorithm to solve an NP-complete problem, you have an algorithm for **every** problem in NP.

An NP-complete problem is a **universal language** for encoding "I'll know it when I see it" problems.

Does one of these exist?

NP Completeness



Did I make the best chess move possible?

NP-Completeness

An NP-complete problem does exist!

Cook-Levin Theorem (1971)

3-SAT is NP-complete

Theorem 1: If a set S of strings is accepted by some nondeterministic Turing machine within polynomial time, then S is P-reducible to {DNF tautologies}.

This sentence (and the proof of it) won Cook the Turing Award.

2-SAT vs. 3-SAT

2-Satisfiability ("2-SAT")

Given: A set of Boolean variables, and a list of requirements, each of the form:

```
variable1==[True/False] || variable2==[True/False]
```

Find: A setting of variables to "true" and "false" so that **all** of the requirements evaluate to "true"

3-Satisfiability ("3-SAT")

Given: A set of Boolean variables, and a list of requirements, each of the form:

```
variable1==[True/False]||variable2==[True/False]||variable3==[True/False]
```

Find: A setting of variables to "true" and "false" so that **all** of the requirements evaluate to "true"

2-SAT vs. 3-SAT

2-Satisfiability ("2-SAT")

Given: A set of Boolean variables, and a list of requirements, each of the form:

```
variable1==[True/False] || variable2==[True/False]
```

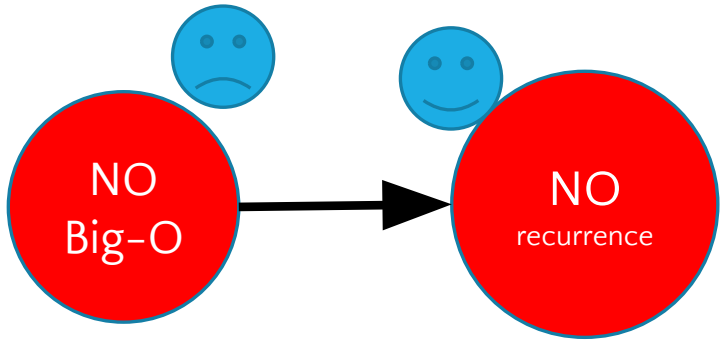
Find: A setting of variables to "true" and "false" so that **all** of the requirements evaluate to "true"

Our first try at 2-SAT (just try all variable settings) would have taken $O(2^Q S)$ time

But we came up with a really clever graph that reduced the time to $O(Q + S)$ time

2-SAT vs. 3-SAT

Can we do the same for 3-SAT?



For 2-SAT we thought we had 2^Q options, but we realized that we didn't have as many choices as we thought - once we made a few choices, out hand was forced and we didn't have to check all possibilities.

3-Satisfiability ("3-SAT")

Given: A set of Boolean variables, and a list of requirements, each of the form:
`variable1==[True/False]||variable2==[True/False]||variable3==[True/False]`
Find: A setting of variables to "true" and "false" so that **all** of the requirements evaluate to "true"

NP-Complete Problems

But Wait! There's more!

Main Theorem. All the problems on the following list are complete.

1. SATISFIABILITY
   COMMENT: By duality, this problem is equivalent to determining whether a disjunctive normal form expression is a tautology.

2. 0-1 INTEGER PROGRAMMING
   INPUT: integer matrix C and integer vector d
   PROPERTY: There exists a 0-1 vector x such that $Cx = d$.

3. CLIQUE
   INPUT: graph G, positive integer k
   PROPERTY: G has a set of k mutually adjacent nodes.

4. SET PACKING
   INPUT: Family of sets $\{S_j\}$, positive integer ℓ
   PROPERTY: $\{S_j\}$ contains ℓ mutually disjoint sets.

5. NODE COVER
   INPUT: graph G', positive integer ℓ
   PROPERTY: There is a set $R \subseteq N'$ such that $|R| \leq \ell$ and every arc is incident with some node in R.

6. SET COVERING
   INPUT: finite family of finite sets $\{S_j\}$, positive integer k
   PROPERTY: There is a subfamily $\{T_h\} \subseteq \{S_j\}$ containing $\leq k$ sets such that $\cup T_h = \cup S_j$.

7. FEEDBACK NODE SET
   INPUT: digraph H, positive integer k
   PROPERTY: There is a set $R \subseteq V$ such that every (directed) cycle of H contains a node in R.

8. FEEDBACK ARC SET
   INPUT: digraph H, positive integer k
   PROPERTY: There is a set $S \subseteq E$ such that every (directed) cycle of H contains an arc in S.

9. DIRECTED HAMILTON CIRCUIT
   INPUT: digraph H
   PROPERTY: H has a directed cycle which includes each node exactly once.

10. UNDIRECTED HAMILTON CIRCUIT
    INPUT: graph G
    PROPERTY: G has a cycle which includes each node exactly once.

11. SATISFIABILITY WITH AT MOST 3 LITERALS PER CLAUSE
    INPUT: Clauses $D_1, D_2, \ldots, D_r$, each consisting of at most 3 literals from the set $\{u_1, u_2, \ldots, u_m\} \cup \{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m\}$
    PROPERTY: The set $\{D_1, D_2, \ldots, D_r\}$ is satisfiable.

12. CHROMATIC NUMBER
    INPUT: graph G, positive integer k
    PROPERTY: There is a function $\phi: N \to Z_k$ such that, if u and v are adjacent, then $\phi(u) \neq \phi(v)$.

13. CLIQUE COVER
    INPUT: graph G', positive integer ℓ
    PROPERTY: N' is the union of ℓ or fewer cliques.

14. EXACT COVER
    INPUT: family $\{S_j\}$ of subsets of a set $\{u_i, i = 1,2,\ldots,t\}$
    PROPERTY: There is a subfamily $\{T_h\} \subseteq \{S_j\}$ such that the sets T_h are disjoint and $\cup T_h = \cup S_j = \{u_i, i = 1,2,\ldots,t\}$.

15. HITTING SET
    INPUT: family $\{U_i\}$ of subsets of $\{s_j, j = 1,2,\ldots,r\}$
    PROPERTY: There is a set W such that, for each i, $|W \cap U_i| = 1$.

16. STEINER TREE
    INPUT: graph G, $R \subseteq N$, weighting function $w: A \to Z$, positive integer k
    PROPERTY: G has a subtree of weight $\leq k$ containing the set of nodes in R.

17. 3-DIMENSIONAL MATCHING
    INPUT: set $U \subseteq T \times T \times T$, where T is a finite set
    PROPERTY: There is a set $W \subseteq U$ such that $|W| = |T|$ and no two elements of W agree in any coordinate.

18. KNAPSACK
    INPUT: $(a_1, a_2, \ldots, a_r, b) \in Z^{n+1}$
    PROPERTY: $\sum a_j x_j = b$ has a 0-1 solution.

19. JOB SEQUENCING
    INPUT: "execution time vector" $(T_1, \ldots, T_p) \in Z^p$,
    "deadline vector" $(D_1, \ldots, D_p) \in Z^p$
    "penalty vector" $(P_1, \ldots, P_p) \in Z^p$
    positive integer k
    PROPERTY: There is a permutation π of $\{1,2,\ldots,p\}$ such that

$$\left(\sum_{j=1}^{p} [\text{if } T_{\pi(1)}+\cdots+T_{\pi(j)} > D_{\pi(j)} \text{ then } P_{\pi(j)} \text{ else } 0]\right) \leq k \ .$$

20. PARTITION
    INPUT: $(c_1, c_2, \ldots, c_s) \in Z^s$
    PROPERTY: There is a set $I \subseteq \{1,2,\ldots,s\}$ such that $\sum_{h \in I} c_h = \sum_{h \notin I} c_h$.

21. MAX CUT
    INPUT: graph G, weighting function $w: A \to Z$, positive integer W
    PROPERTY: There is a set $S \subseteq N$ such that

$$\sum_{\substack{\{u,v\} \in A \\ u \in S \\ v \notin S}} w(\{u,v\}) \geq W \ .$$

Karp's Theorem (1972)

A lot of problems are NP-complete

NP-Complete Problems

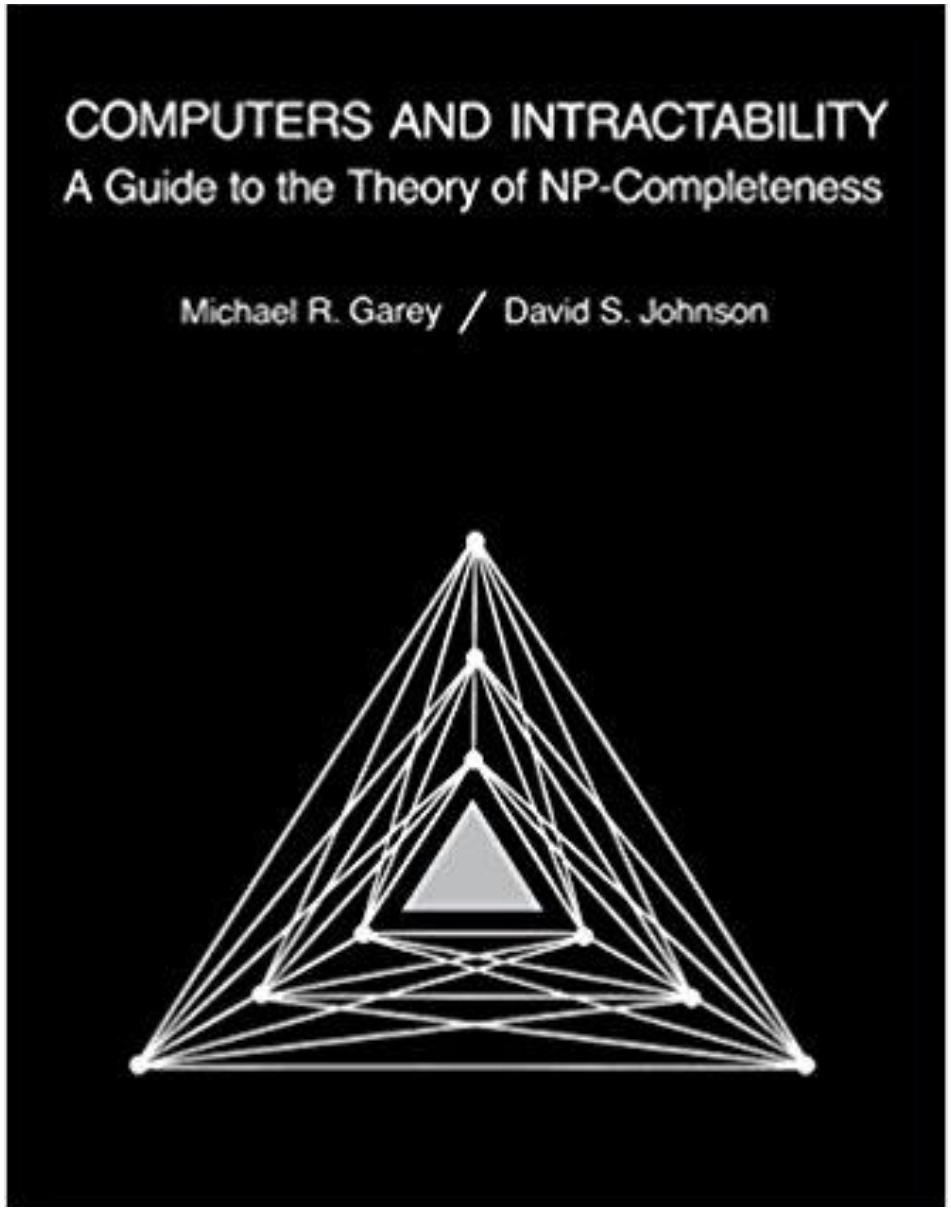
But Wait! There's more!

By 1979, at least 300 problems had been proven NP-complete.

Garey and Johnson put a list of all the NP-complete problems they could find in this textbook.

Took almost 100 pages to just list them all.

No one has made a comprehensive list since.



NP-Complete Problems

But Wait! There's more!

In the last month, mathematicians and computer scientists have put papers on the arXiv claiming to show (at least) 25 more problems are NP-complete.

There are literally thousands of NP-complete problems known.

And some of them look weirdly similar to problems we've already studied.

Examples

There are literally thousands of NP-complete problems.
And some of them look weirdly similar to problems we do know efficient algorithms for.

In P

Short Path

Given a directed graph, report if there is a path from s to t of length at most *k*

NP-Complete

Long Path

Given a directed graph, report if there is a path from s to t of length at least *k*

Examples

In P

Light Spanning Tree

Given a weighted graph, find a spanning tree (a set of edges that connect all vertices) of weight at most *k*

NP-Complete

Traveling Salesperson

Given a weighted graph, find a tour (a walk that visits every vertex and returns to its start) of minimum weight

The electric company just needs a greedy algorithm to lay its wires. Amazon doesn't know a way to optimally route its delivery trucks.

Dealing with NP-Completeness

Option 1: Maybe it's a special case we understand

Maybe you don't need to solve the general problem, just a special case

Option 2: Maybe it's a special case we *don't* understand (yet)

There are algorithms that are known to run quickly on "nice" instances. Maybe your problem has one of those.

One approach: Turn your problem into a SAT instance, find a solver and cross your fingers.

Dealing with NP-Completeness

Option 3: Approximation Algorithms

You might not be able to get an exact answer, but you might be able to get close.

Optimization version of Traveling Salesperson
Given a weighted graph, find a tour (a walk that visits every vertex and returns to its start) of weight at most *k*.

Algorithm:
Find a minimum spanning tree.
Have the tour follow the visitation order of a DFS of the spanning tree.
Theorem: This tour is at most twice as long as the best one.

Why should you care about P vs. NP

Most computer scientists are convinced that P≠NP.

Why should you care about this problem?

It's your chance for:

- $1,000,000. The Clay Mathematics Institute will give $1,000,000 to whoever solves P vs. NP (or any of the 5 remaining problems they listed)
- To get a Turing Award

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- To get ~~a Turing Award~~ the Turing Award named after you

Why Should You Care if P=NP?

Suppose P=NP.

Specifically that we found a genuinely in-practice efficient algorithm for an NP-complete problem. What would you do?

- $1,000,000 from the Clay Math Institute obviously, but what's next?

Why Should You Care if P=NP?

We found a genuinely in-practice efficient algorithm for an NP-complete problem. What would you do?

- Another $5,000,000 from the Clay Math Institute
- Put mathematicians out of work.
- Decrypt (essentially) all current internet communication.
- No more secure online shopping or online banking or online messaging…or online *anything*.
- Cure cancer with efficient protein folding

A world where P=NP is a very very different place from the world we live in now.

Why Should You Care if P≠NP?

We already expect P ≠ NP. Why should you care when we finally prove it?

P ≠ NP says something fundamental about the universe.

For some questions there is not a clever way to find the right answer

- Even though you'll know it when you see it
- Some problems require "creative leaps" to find a solution that cannot be programmed

There is actually a way to obscure information, so it cannot be found quickly no matter how clever you are.

Why Should You Care if P≠NP?

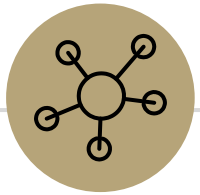
To prove P≠NP we need to better understand the differences between problems.

- Why do some problems allow easy solutions and others don't?
- What is the structure of these problems?

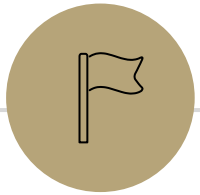
We don't care about P vs NP just because it has a huge effect about what the world looks like.

We will learn a lot about computation along the way.

If P = NP, then the world would be a profoundly different place than we usually assume it to be. There would be no special value in "creative leaps", no fundamental gap between solving a problem and recognizing the solution once it's found. Everyone who could appreciate a symphony would be Mozart. Everyone who could follow a step by step argument would be Gauss" -Scott Aaronson, MIT complexity researcher



Questions?



That’s all!