Lecture 19: Disjoint Sets

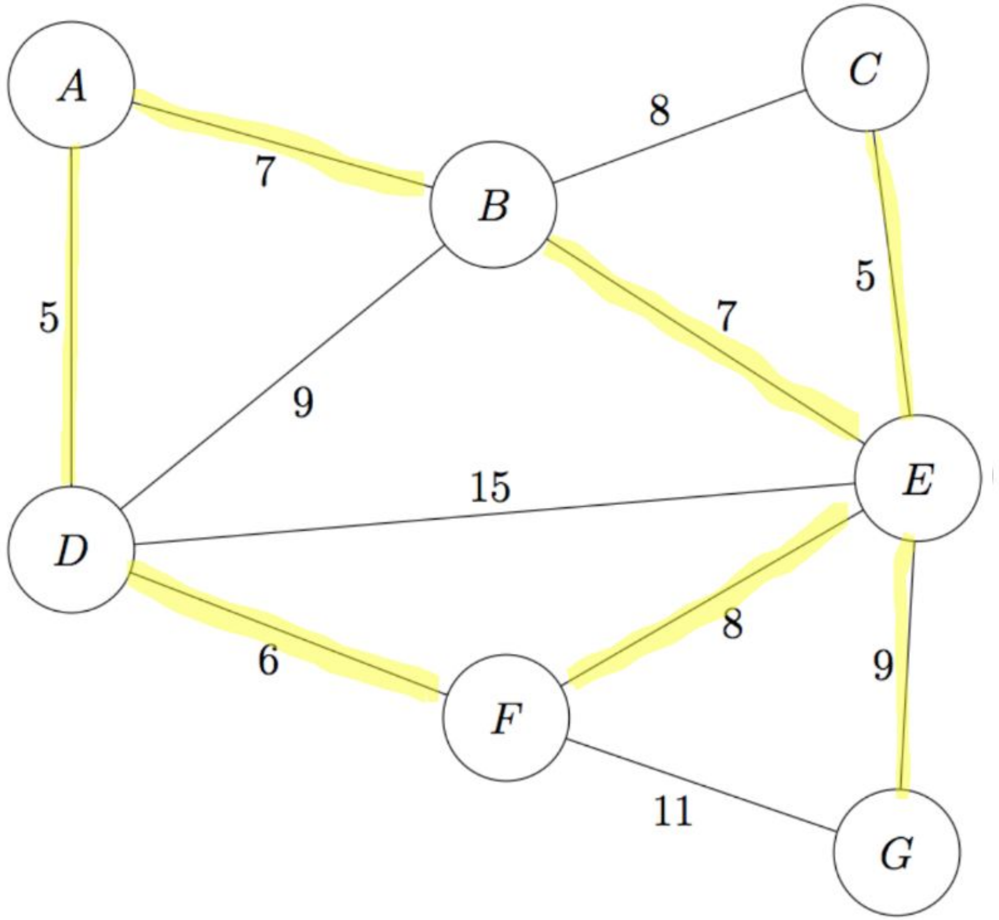
CSE 373: Data Structures and Algorithms

Warmup

Run Kruskal's algorithm on the following graph to find the MST (minimum spanning tree) of the graph below.
Below is the provided pseudocode for Kruksal's algorithm to choose all the edges.

Slido Event #4005803
https://app.sli.do/event/6wH7a1g9cB6kLbzgzuJNMh

```
KruskalMST(Graph G)
   initialize each vertex to be an independent
component
   sort the edges by weight
   foreach(edge (u, v) in sorted order){
      if(u and v are in different components){
         add (u,v) to the MST
         update u and v to be in the same component
      }
   }
```



Announcements

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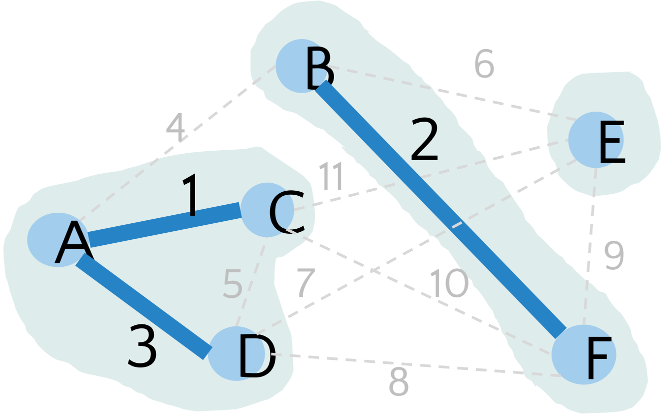
Disjoint Sets

Selecting an ADT



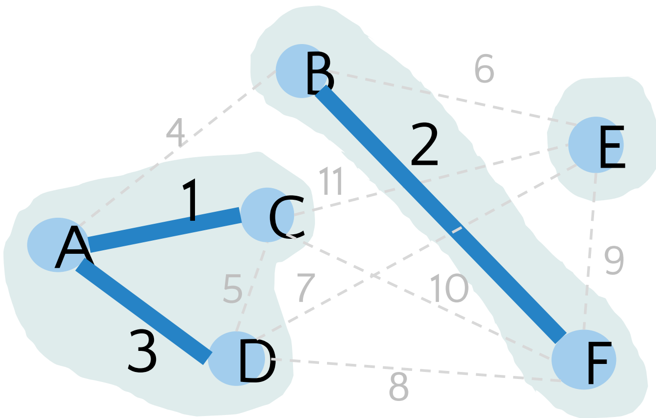
Kruskal's needs to **find** what MST a vertex belongs to, and **union** those MSTs together

- Our existing ADTs don't lend themselves well to "unioning" two sets…
- Let's define a new one!

```
kruskalMST(G graph)
  Set(?) msts; Set finalMST;
  initialize msts with each vertex as single-element MST
  sort all edges by weight (smallest to largest)

  for each edge (u,v) in ascending order:
    uMST = msts.find(u)
    vMST = msts.find(v)
    if (uMST != vMST):
      finalMST.add(edge (u, v))
      msts.union(uMST, vMST)
```

Disjoint Sets ADT (aka "Union-Find")



Kruskal's will use a Disjoint Sets ADT under the hood

- Conceptually, a single instance of this ADT contains a "family" of sets that are disjoint (no element belongs to multiple sets)

```
kruskalMST(G graph)
  DisjointSets<V> msts; Set finalMST;
  initialize msts with each vertex as single-element MST
  sort all edges by weight (smallest to largest)

  for each edge (u,v) in ascending order:
    uMST = msts.find(u)
    vMST = msts.find(v)
    if (uMST != vMST):
      finalMST.add(edge (u, v))
      msts.union(uMST, vMST);
```

DISJOINT SETS ADT

State

Family of Sets
- disjoint: no shared elements
- each set has a representative (either a member or a unique ID)

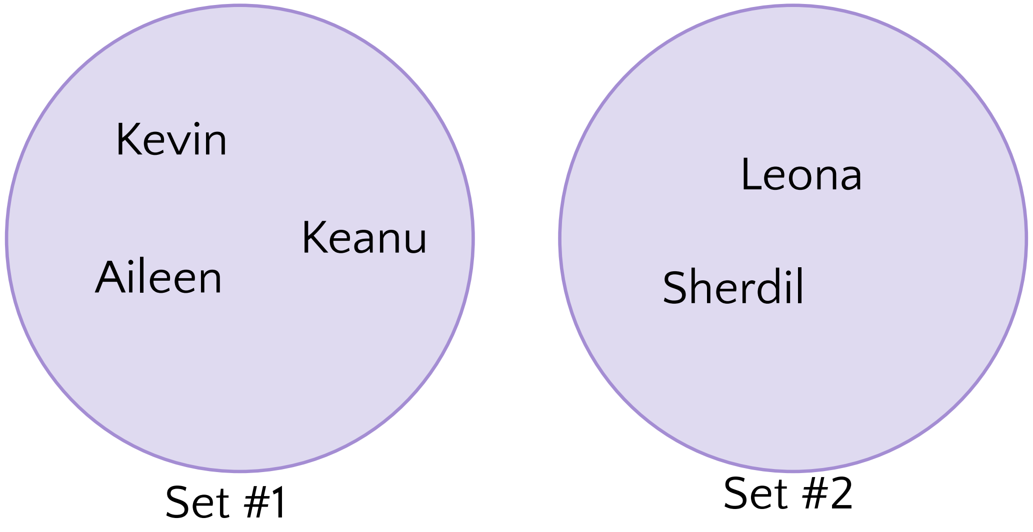
Behavior

<u>makeSet(value)</u> – new set with value as only member (and representative)
<u>find(value)</u> – return representative of the set containing value
<u>union(x, y)</u> – combine sets containing x and y into one set with all elements, choose single new representative

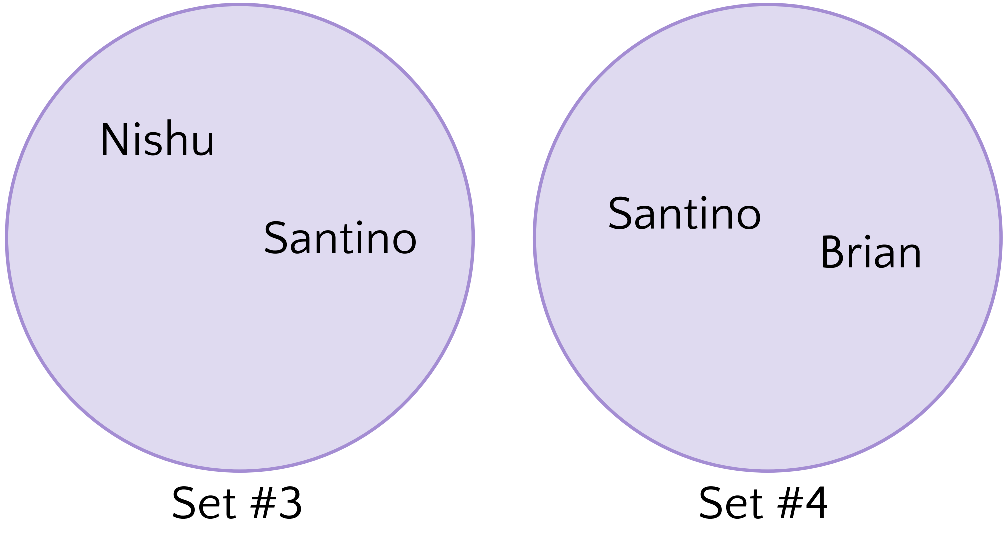
Disjoint Sets in mathematics

"In mathematics, two **sets** are said to be **disjoint sets** if they have no element in common." - Wikipedia

disjoint = not overlapping



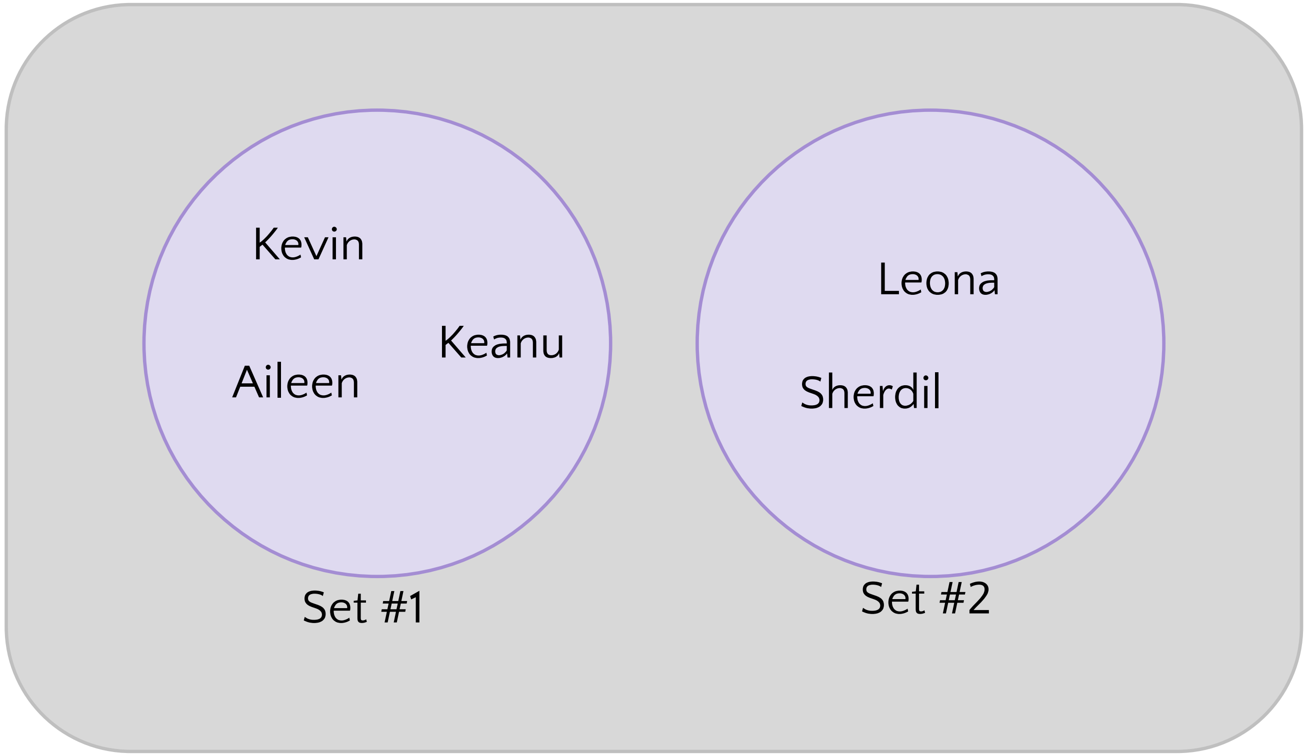
These two sets are disjoint sets



These two sets are not disjoint sets

Disjoint Sets in computer science

In computer science, disjointsets can refer to this ADT/data structure that keeps track of the multiple “mini” sets that are disjoint (confusing naming, I know)



This overall grey blob thing is the actual disjoint sets, and it’s keeping track of any number of mini-sets, which are all disjoint (the mini sets have no overlapping values).

Note: this might feel really different than ADTs we’ve run into before. The ADTs we’ve seen before (dictionaries, lists, sets, etc.) just store values directly.
But the Disjoint Set ADT is particularly interested in letting you group your values into sets and keep track of which particular set your values are in.

DisjointSets ADT methods

Just 3 methods (and makeSet is pretty simple!)

- findSet(value)
- union(valueA, valueB)
- makeSet(value)

findSet(value)

findSet(value) returns some ID for which particular set the value is in. For Disjoint Sets, we often call this the **representative** (as it's a value that represents the whole set).

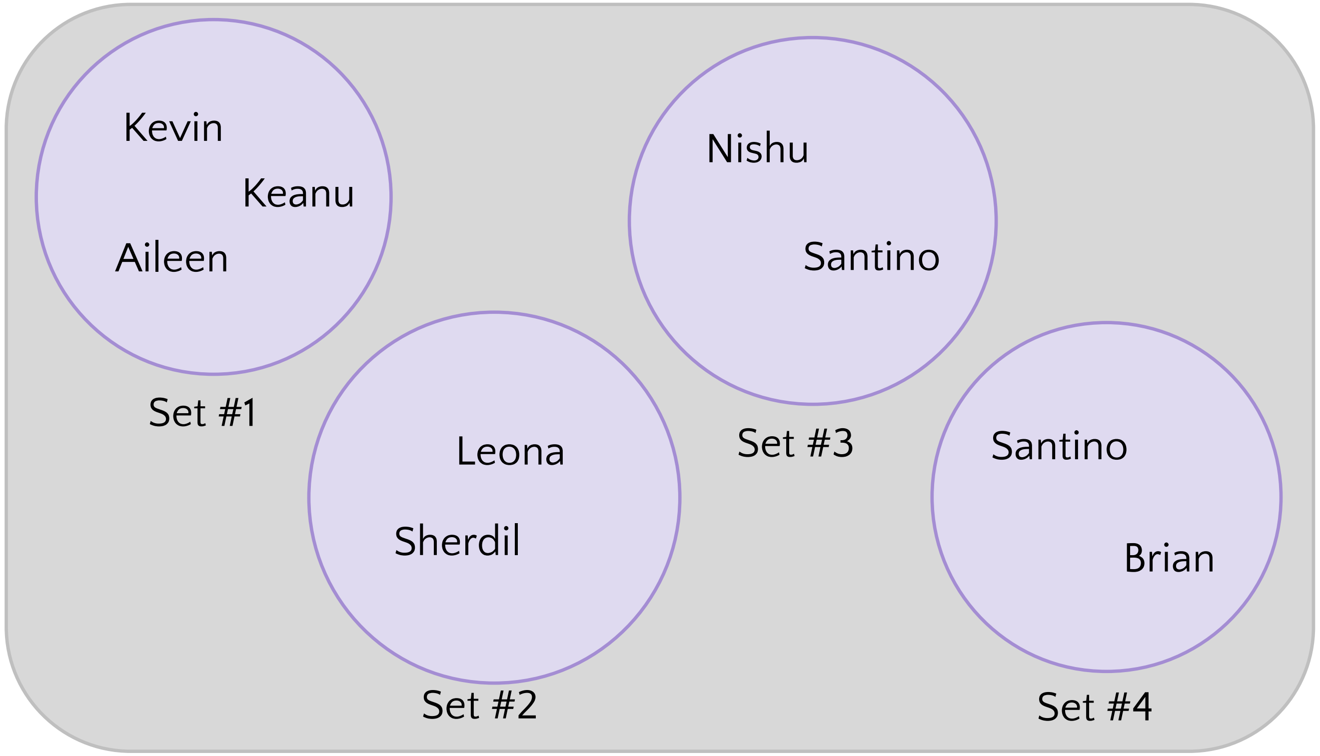
Examples:

findSet(Brian) 3

findSet(Sherdil) 2

findSet(Velocity) 2

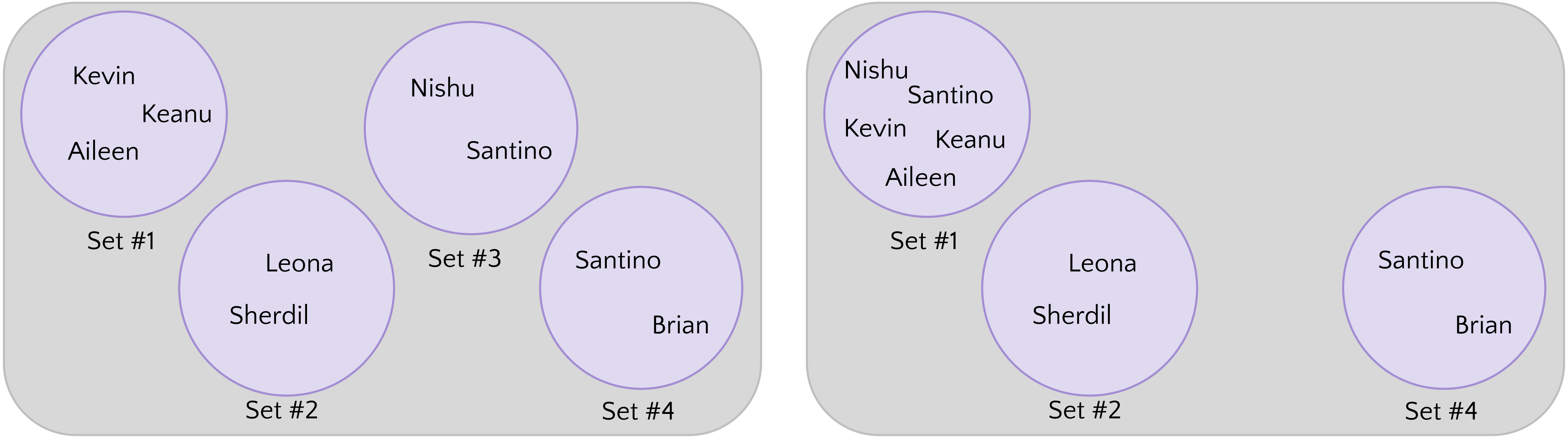
findSet(Kevin) == findSet(Aileen) true



union(valueA, valueB)

union(valueA, valueB) merges the set that A is in with the set that B is in.
(basically add the two sets together into one)

Example: union(Kevin, Nishu)



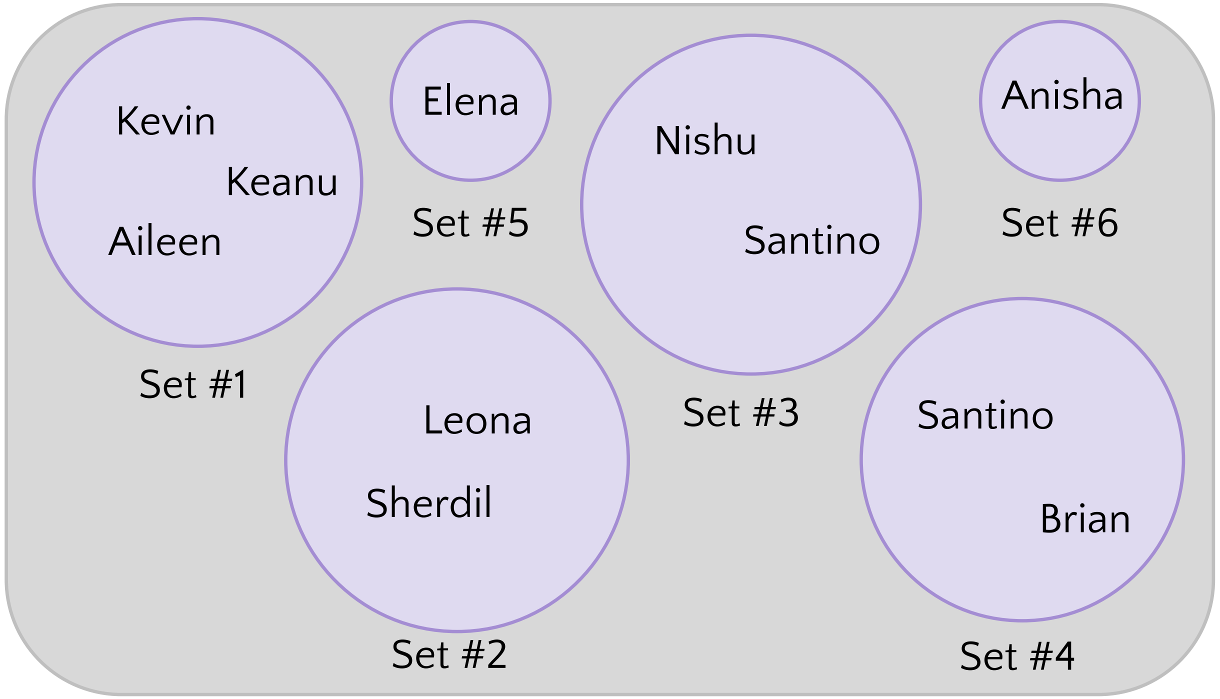
makeSet(value)

makeSet(value) makes a new mini set that just has the value parameter in it.

Examples:

makeSet(Elena)

makeSet(Anish)



Disjoint Sets ADT Summary

Disjoint-Sets ADT

state

Set of Sets

- **Mini sets are disjoint:** Elements must be unique across mini sets
- No required order
- Each set has id/representative

behavior

makeSet(value) – creates a new set within the disjoint set where the only member is the value. Picks id/representative for set

findSet(value) – looks up the set containing the value, returns id/representative/ of that set

union(x, y) – looks up set containing x and set containing y, combines two sets into one. All of the values of one set are added to the other, and the now empty set goes away.

New ADT

Set ADT

state

Set of elements
- Elements must be unique!
- No required order

Count of Elements

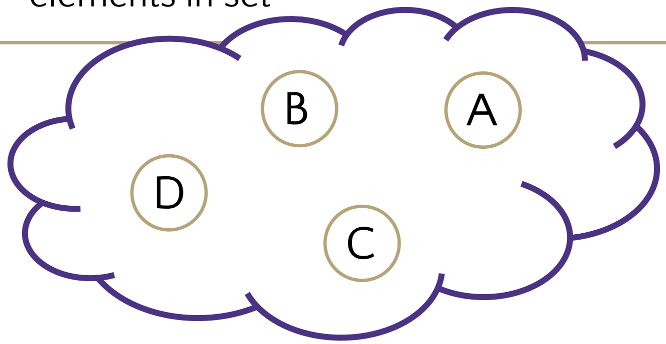
behavior

create(x) – creates a new set with a single member, x

add(x) – adds x into set if it is unique, otherwise add is ignored

remove(x) – removes x from set

size() – returns current number of elements in set



Disjoint-Set ADT

state

Set of Sets
- **Disjoint:** Elements must be unique across sets
- No required order
- Each set has representative

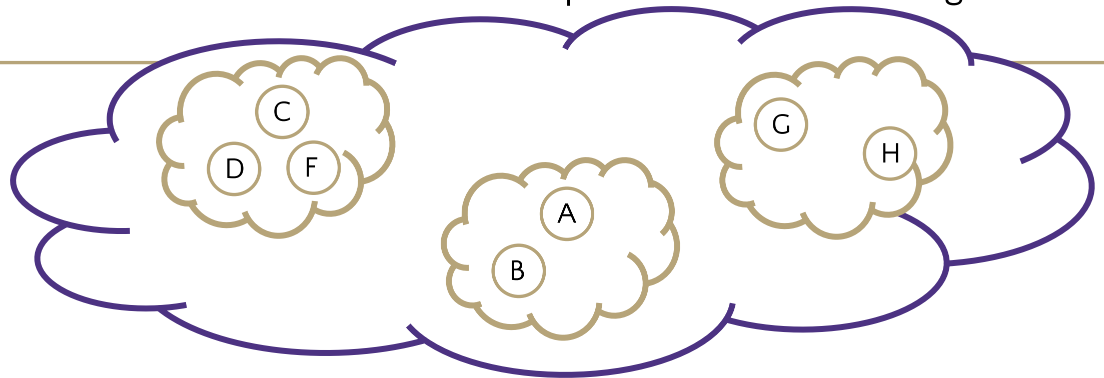
Count of Sets

behavior

makeSet(x) – creates a new set within the disjoint set where the only member is x. Picks representative for set

findSet(x) – looks up the set containing element x, returns representative of that set

union(x, y) – looks up set containing x and set containing y, combines two sets into one. Picks new representative for resulting set



Example

new()

makeSet(a)

makeSet(b)

makeSet(c)

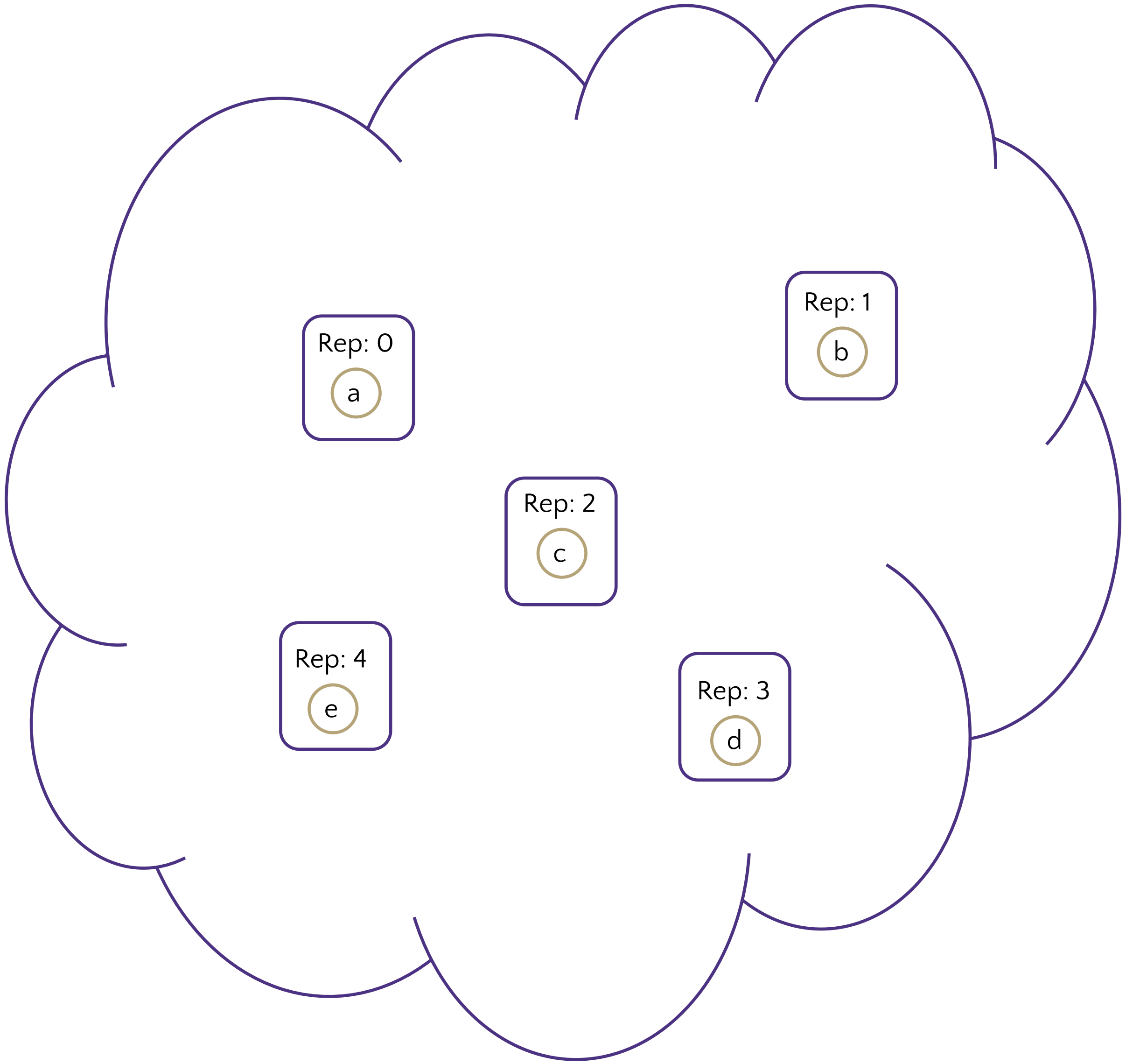
makeSet(d)

makeSet(e)

findSet(a)

findSet(d)

union(a, c)



Example

new()

makeSet(a)

makeSet(b)

makeSet(c)

makeSet(d)

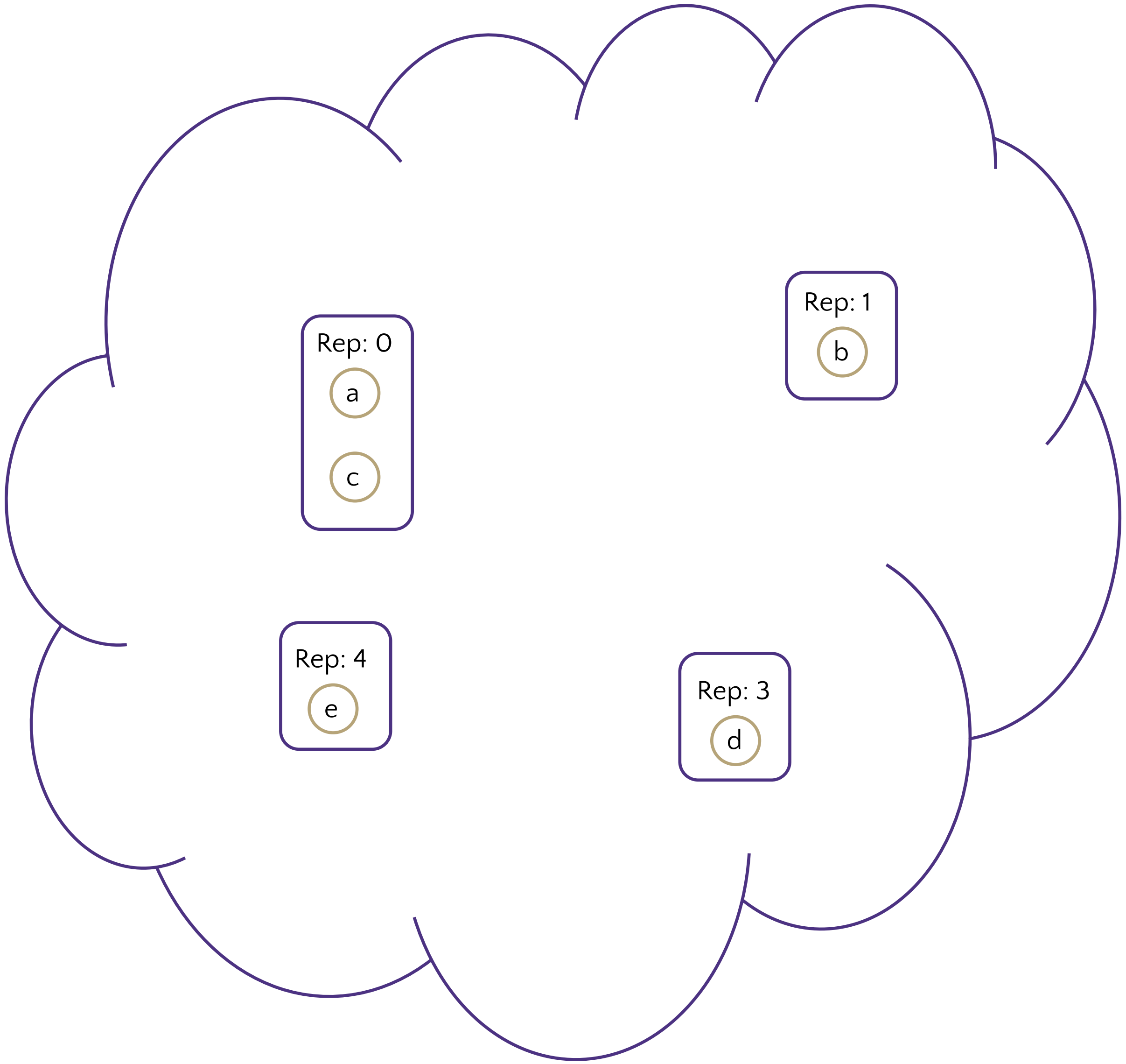
makeSet(e)

findSet(a)

findSet(d)

union(a, c)

union(b, d)



Example

new()

makeSet(a)

makeSet(b)

makeSet(c)

makeSet(d)

makeSet(e)

findSet(a)

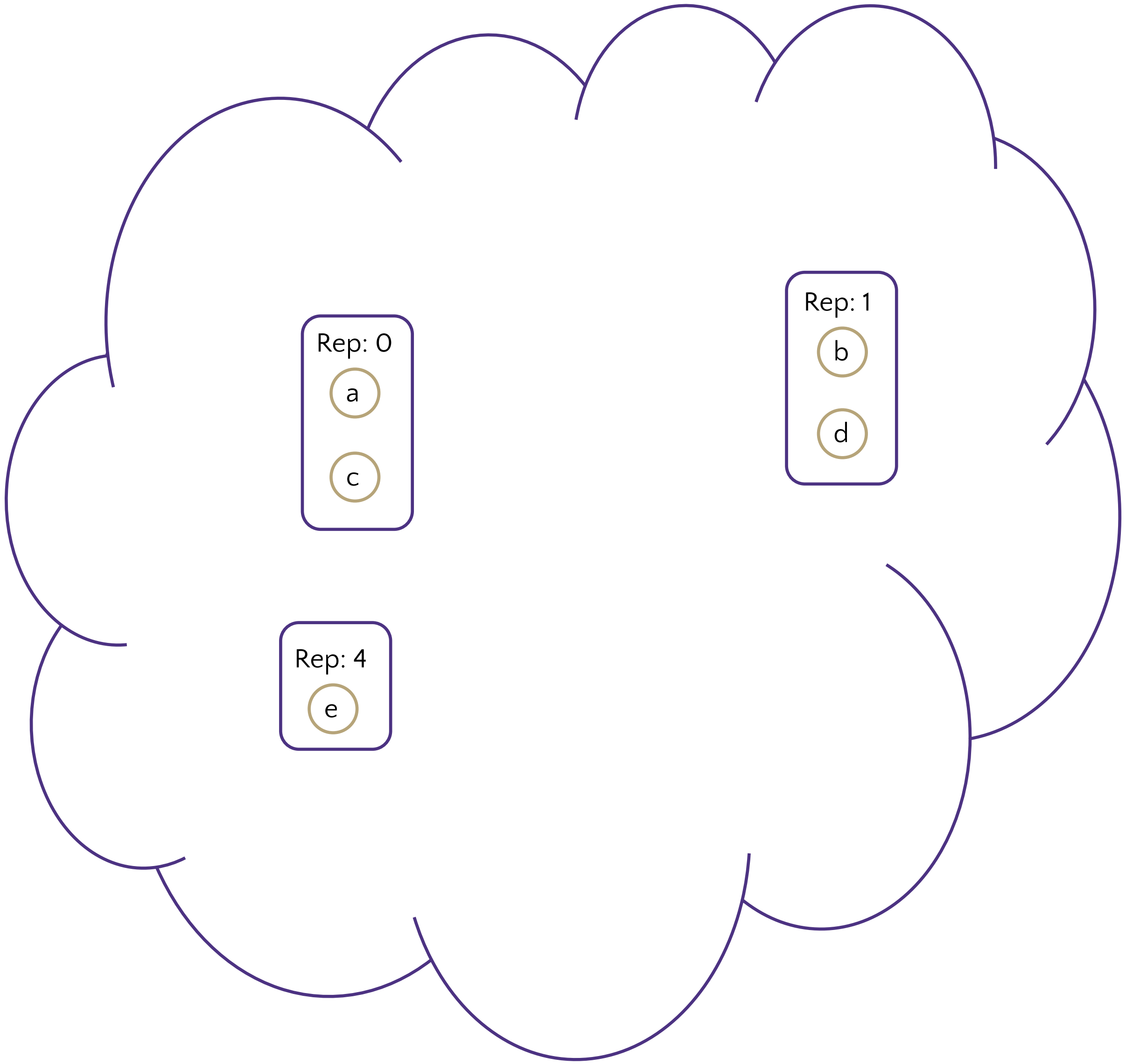
findSet(d)

union(a, c)

union(b, d)

findSet(a) == findSet(c) true

findSet(a) == findSet(d) false



Questions?

That's all!

Implementation

Disjoint-Set ADT

state

Set of Sets
- **Disjoint:** Elements must be unique across sets
- No required order
- Each set has representative

Count of Sets

behavior

makeSet(x) – creates a new set within the disjoint set where the only member is x. Picks representative for set

findSet(x) – looks up the set containing element x, returns representative of that set

union(x, y) – looks up set containing x and set containing y, combines two sets into one. Picks new representative for resulting set

TreeDisjointSet<E>

state

```
Collection<TreeSet> forest
Dictionary<NodeValues, NodeLocations> nodeInventory
```

behavior

```
makeSet(x)-create a new tree of size 1 and add to our forest

findSet(x)-locates node with x and moves up tree to find root

union(x, y)-append tree with y as a child of tree with x
```

TreeSet<E>

state

```
SetNode overallRoot
```

behavior

```
TreeSet(x)
add(x)
remove(x, y)
getRep()-returns data of overallRoot
```

SetNode<E>

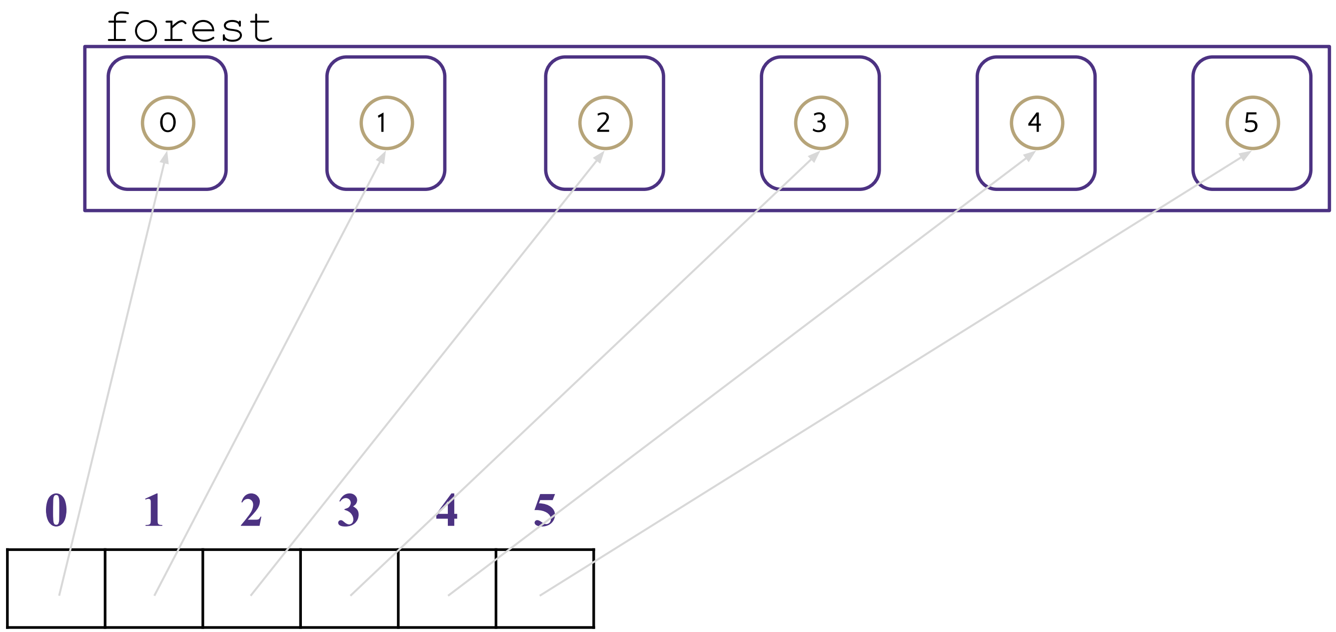
state

```
E data
Collection<SetNode> children
```

behavior

```
SetNode(x)
addChild(x)
removeChild(x, y)
```

Implement makeSet(x)




```
makeSet(0)
makeSet(1)
makeSet(2)
makeSet(3)
makeSet(4)
makeSet(5)
```

TreeDisjointSet<E>

state

```
Collection<TreeSet> forest
Dictionary<NodeValues,
NodeLocations> nodeInventory
```

behavior

`makeSet(x)`-create a new tree of size 1 and add to our forest

`findSet(x)`-locates node with x and moves up tree to find root

`union(x, y)`-append tree with y as a child of tree with x

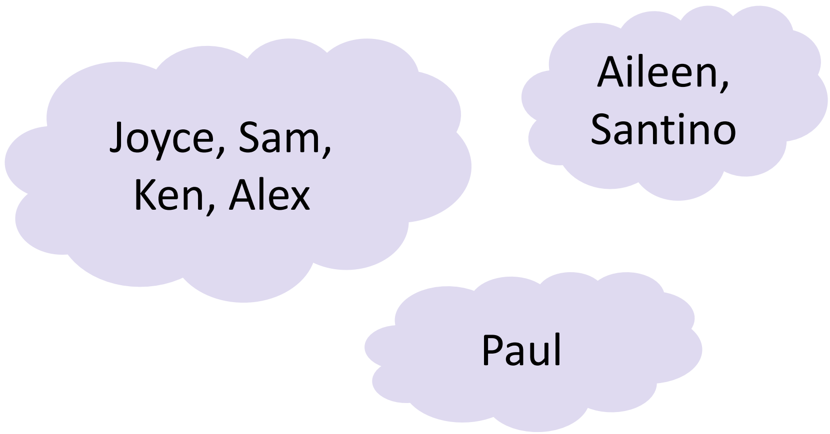
Worst case runtime?

O(1)

QuickUnion Data Structure

Fundamental idea:

- QuickFind tracks each element's ID
- QuickUnion tracks each element's *parent*. Only the root has an ID!
  - Each set becomes tree-like, but something slightly different called an **up-tree**: store pointers from children to parents!



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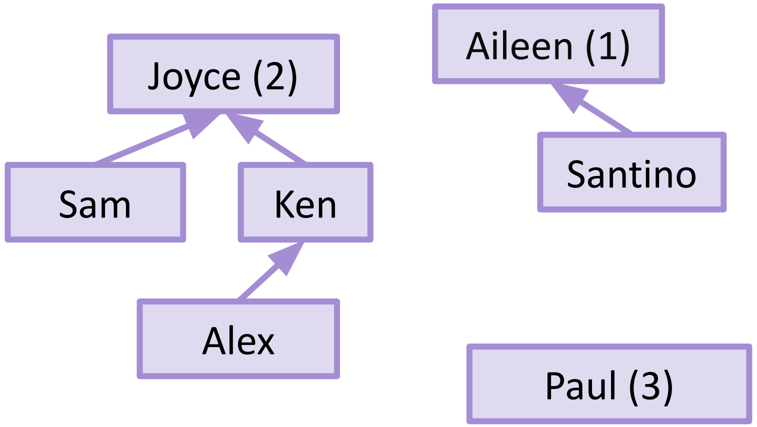


Abstract Idea of "Disjoint Sets"

Implementation using QuickUnion

QuickUnion: Find

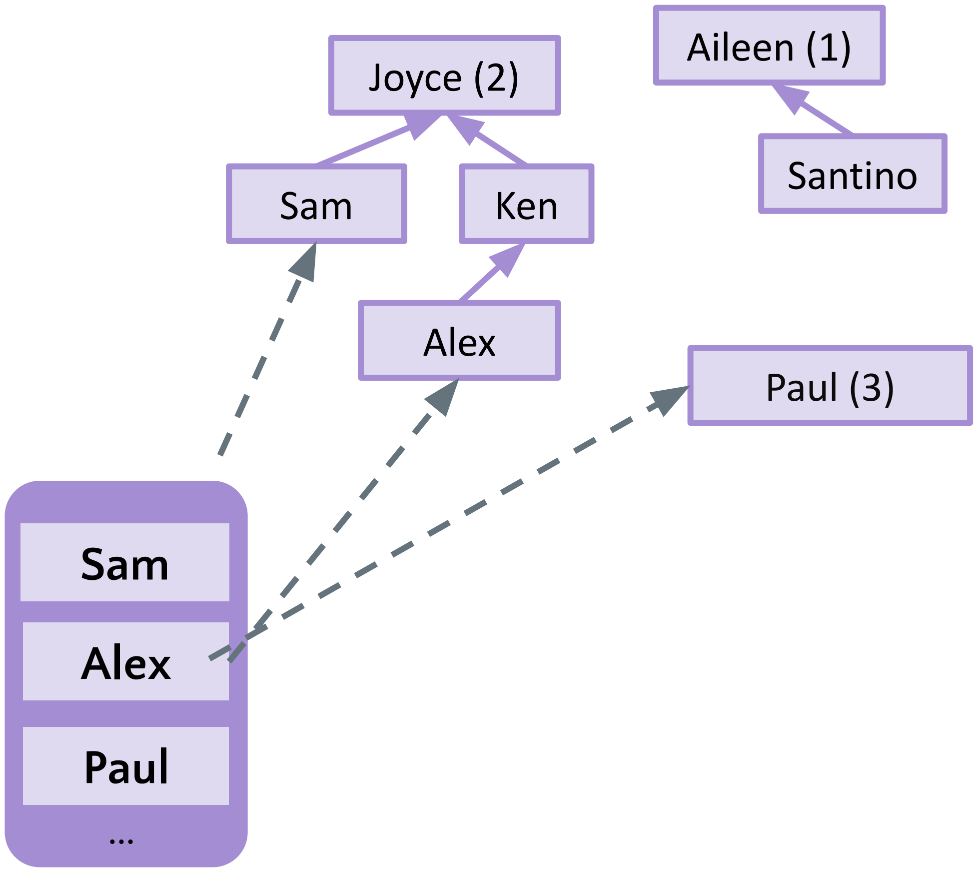
```
find(Ken):
  jump to Ken node
  travel upward until root
  return ID
```

Key idea: can travel upward from any node to find its representative ID

How do we jump to a node quickly?

- *Also* store a map from value to its node (Omitted in future slides)

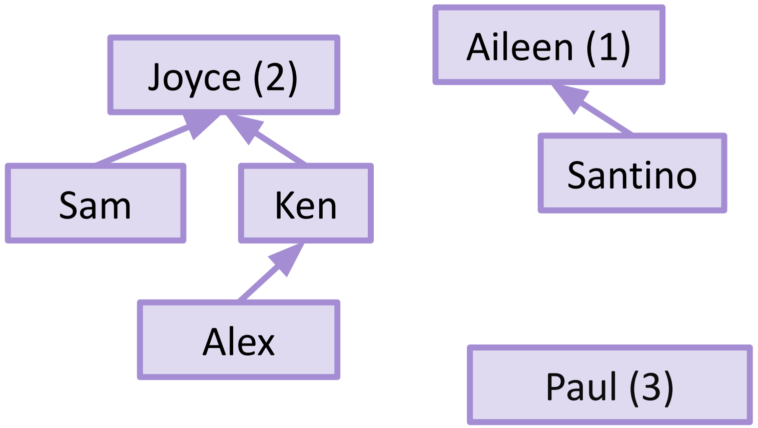
```
find(Santino) -> 1
find(Ken) -> 2
find(Santino) != find(Ken)
find(Santino) == find(Aileen)
```



QuickUnion: Union

Key idea: easy to simply rearrange pointers to union entire trees together!

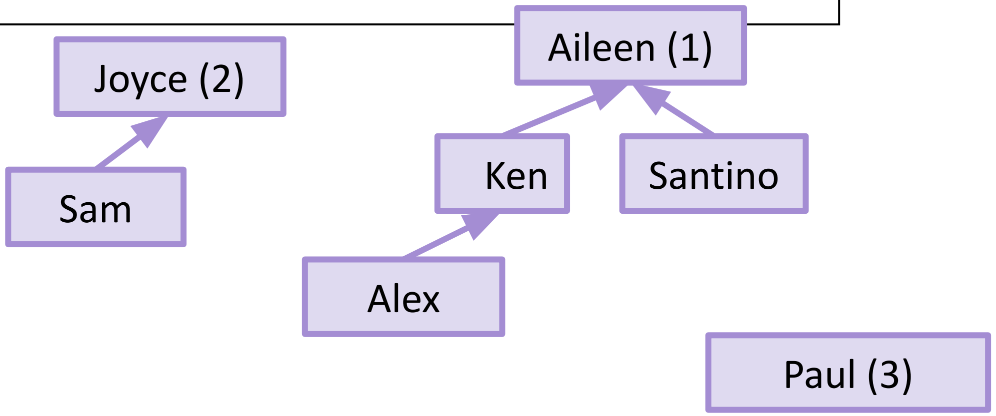
Which of these implementations would you prefer?

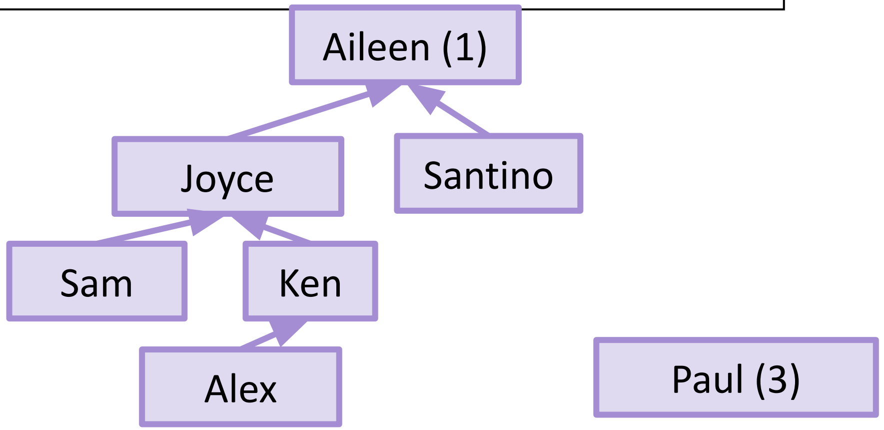



```
union(Ken, Santino):
  rootS = find(Santino)
  set Ken to point to rootS
```

```
union(Ken, Santino):
  rootK = find(Ken)
  rootS = find(Santino)
  set rootK to point to rootS
```

RESULT:



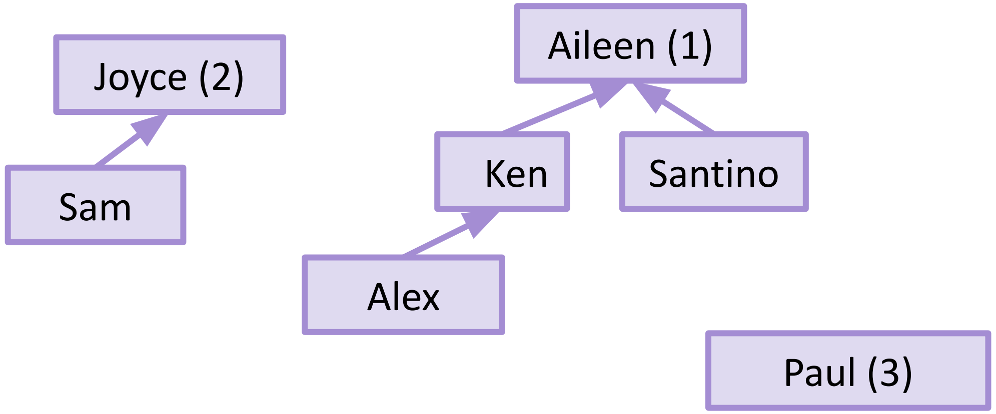


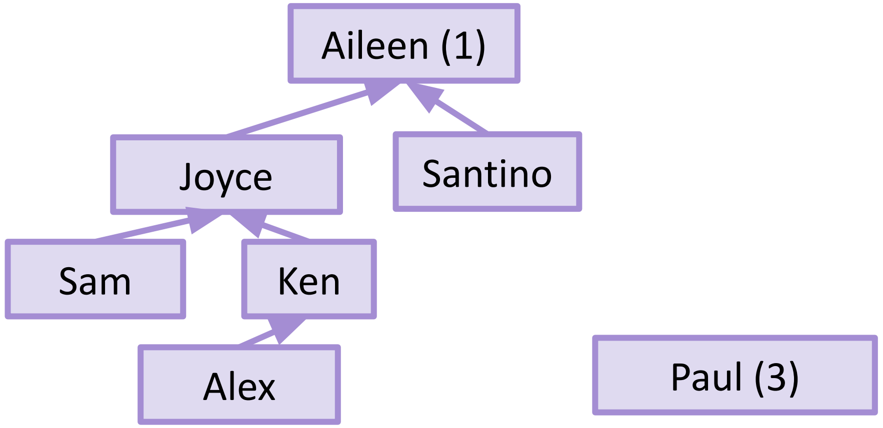
QuickUnion: Union

```
union(Ken, Santino):
  rootS = find(Santino)
  set Ken to point to rootS
```

```
union(Ken, Santino):
  rootK = find(Ken)
  rootS = find(Santino)
  set rootK to point to rootS
```

RESULT:



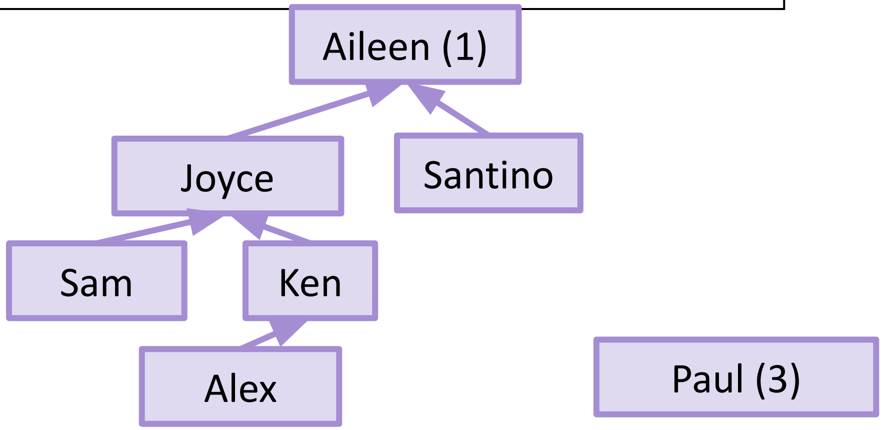


We prefer the right implementation because by changing just the root, we effectively pull the entire tree into the new set!

- If we change the first node instead, we have to do more work for the rest of the old tree
- A rare example of constant time work manipulating a factor of n elements

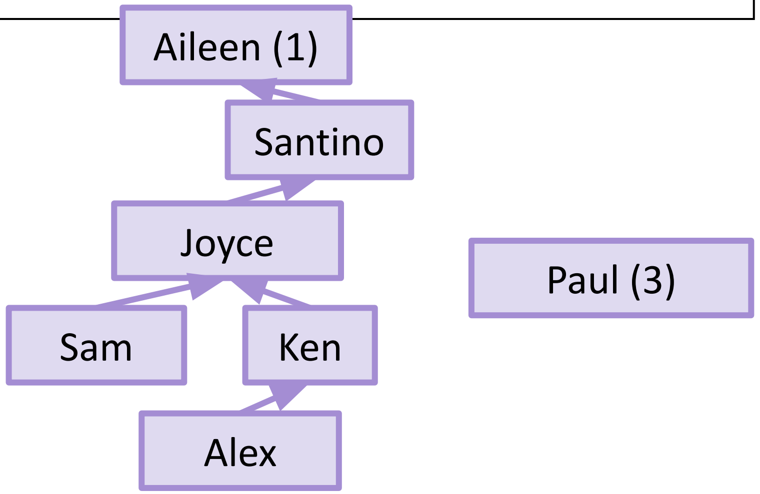
QuickUnion: Why bother with the second root?

```
union(Ken, Santino):
  rootK = find(Ken)
  rootS = find(Santino)
  set rootK to point to rootS
```



Why not just use:

```
union(Ken, Santino):
  rootK = find(Ken)
  set rootK to point to Santino
```



Key idea: will help minimize runtime for future find() calls if we keep the height of the tree short!

- Pointing directly to the second element would make the tree taller

QuickUnion: Checking in on those runtimes

	Maps to Sets	QuickFind	QuickUnion
`makeSet(value)`	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
`findSet(value)`	$\Theta(n)$	$\Theta(1)$	$\Theta(n)$
`union(x, y)`	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$

Only if we discount the runtime from union's calls to find! Otherwise, $\Theta(n)$.

- However, for Kruskal's not a bad assumption: we only ever call union with roots anyway

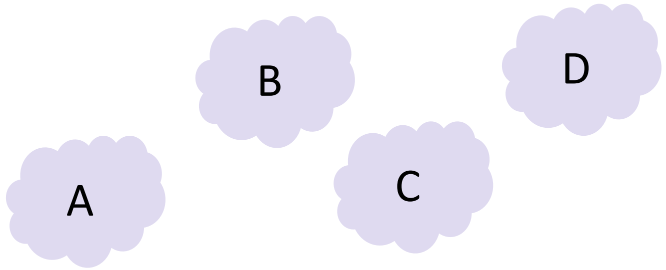
```
union(A, B):
  rootA = find(A)
  rootB = find(B)
  set smallerRoot to point to largerRoot
```

```
kruskalMST(G graph)
  DisjointSets<V> msts; Set finalMST;
  initialize msts with each vertex as single-element MST
  sort all edges by weight (smallest to largest)

  for each edge (u,v) in ascending order:
    uMST = msts.find(u)
    vMST = msts.find(v)
    if (uMST != vMST):
      finalMST.add(edge (u, v))
      msts.union(uMST, vMST);
```

QuickUnion: Let's Build a Worst Case

Even with the "use-the-roots" implementation of union, try to come up with a series of calls to union that would create a worst-case runtime for find on these Disjoint Sets:

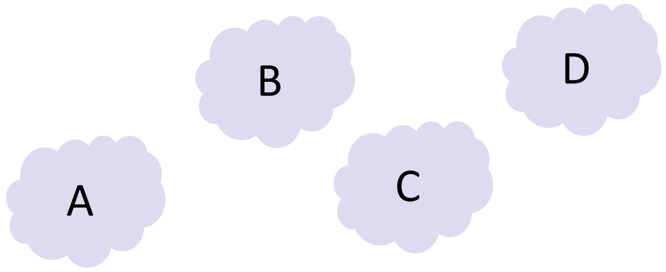



```
find(A):
  jump to A node
  travel upward until root
  return ID
```

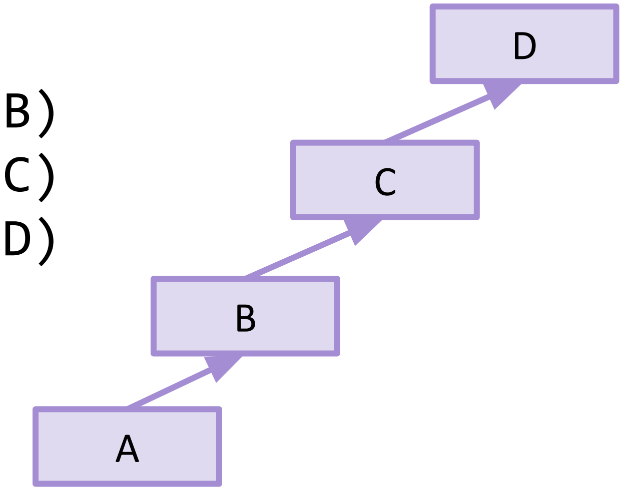
```
union(A, B):
  rootA = find(A)
  rootB = find(B)
  set rootA to point to rootB
```

QuickUnion: Let's Build a Worst Case

Even with the "use-the-roots" implementation of union, try to come up with a series of calls to union that would create a worst-case runtime for find on these Disjoint Sets:




```
union(A, B)
union(B, C)
union(C, D)
find(A)
```



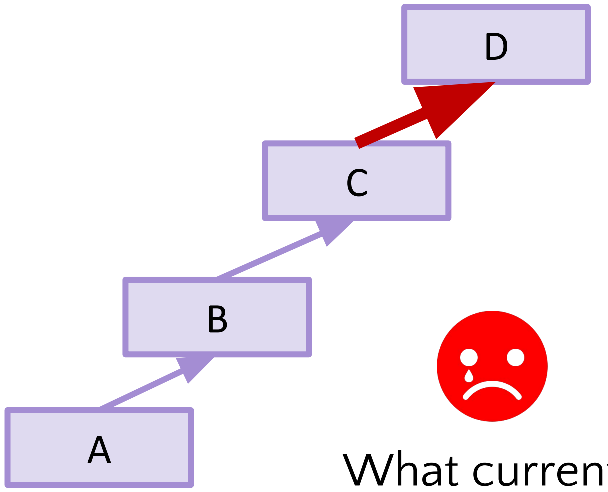

```
find(A):
  jump to A node
  travel upward until root
  return ID
```

```
union(A, B):
  rootA = find(A)
  rootB = find(B)
  set rootA to point to rootB
```

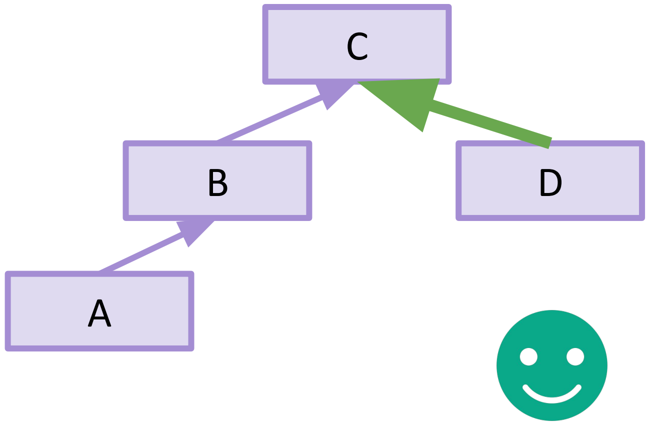
Analyzing the QuickUnion Worst Case

- How did we get a degenerate tree?
  - Even though pointing a root to a root usually helps with this, we can still get a degenerate tree **if we put the root of a large tree under the root of a small tree.**
  - In QuickUnion, rootA always goes under rootB
    - But what if we could ensure the smaller tree went under the larger tree?

`union(C, D)`



What currently happens



What would help avoid degenerate tree

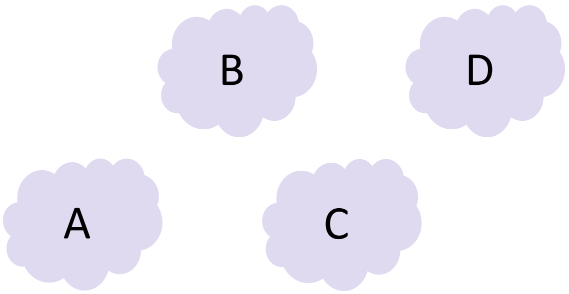
WeightedQuickUnion

Goal: Always pick the smaller tree to go under the larger tree

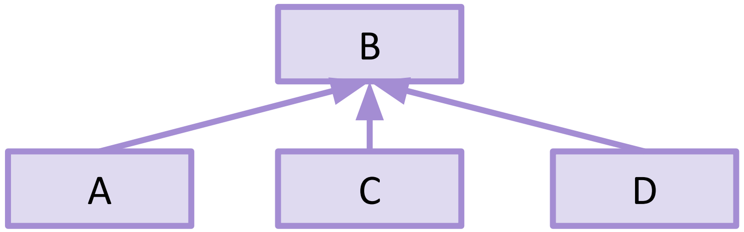
Implementation: Store the number of nodes (or “weight”) of each tree in the root

- Constant-time lookup instead of having to traverse the entire tree to count

```
union(A, B)
union(B, C)
union(C, D)
find(A)
```



Now what happens?



Perfect! Best runtime we can get.

```
union(A, B):
  rootA = find(A)
  rootB = find(B)
  put lighter root under heavier root
```

WeightedQuickUnion: Performance

union()'s runtime is still dependent on find()'s runtime, which is a function of the tree's height

What's the worst-case height for WeightedQuickUnion?

```
union(A, B):
  rootA = find(A)
  rootB = find(B)
  put lighter root under heavier root
```

WeightedQuickUnion: Performance

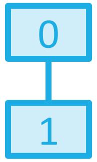
Consider the worst case where the tree height grows as fast as possible

N	H
1	0

WeightedQuickUnion: Performance

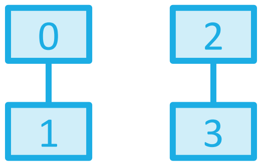
Consider the worst case where the tree height grows as fast as possible

N	H
1	0
2	1



WeightedQuickUnion: Performance

Consider the worst case where the tree height grows as fast as possible

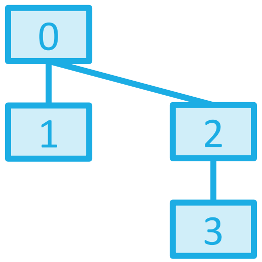


N	H
1	0
2	1
4	?

WeightedQuickUnion: Performance

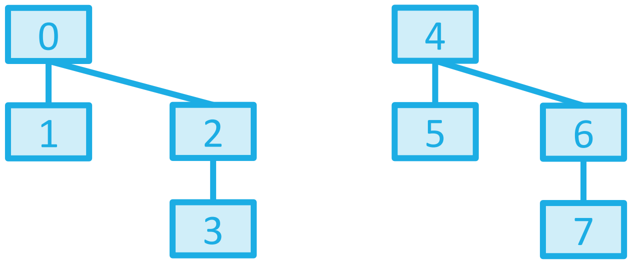
Consider the worst case where the tree height grows as fast as possible

N	H
1	0
2	1
4	2



WeightedQuickUnion: Performance

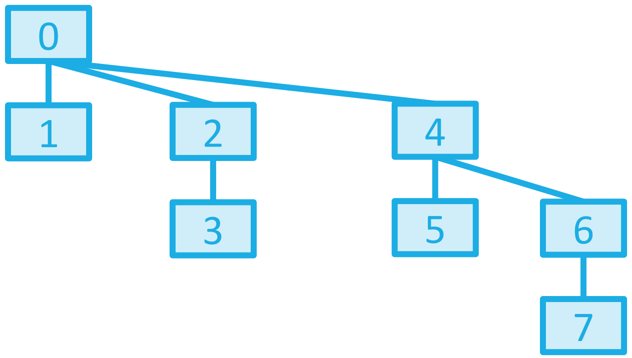
Consider the worst case where the tree height grows as fast as possible



N	H
1	0
2	1
4	2
8	?

WeightedQuickUnion: Performance

Consider the worst case where the tree height grows as fast as possible



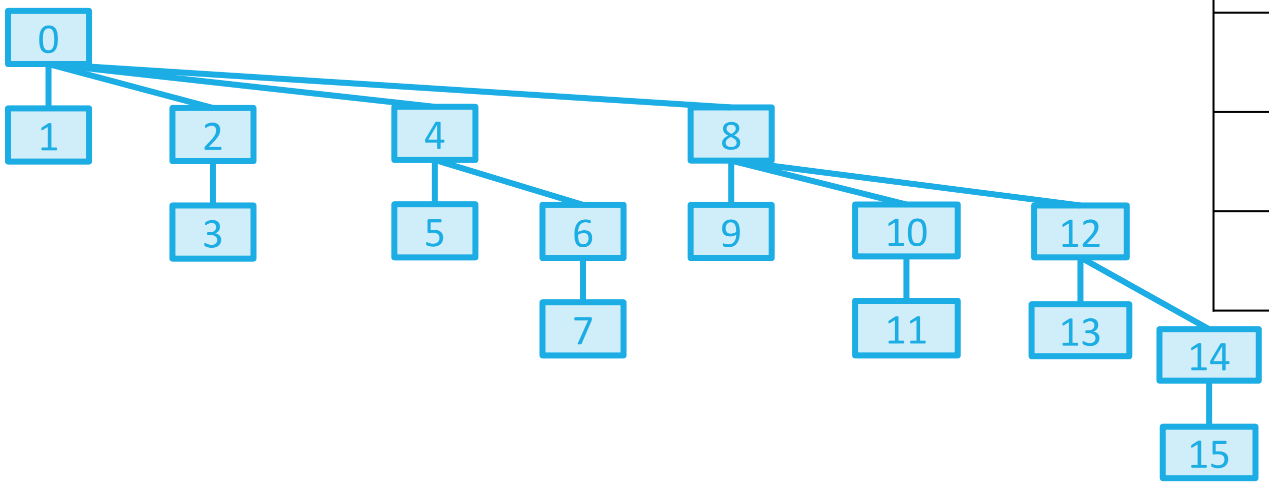
N	H
1	0
2	1
4	2
8	3

WeightedQuickUnion: Performance

Consider the worst case where the tree height grows as fast as possible

Worst case tree height is Θ(log N)

N	H
1	0
2	1
4	2
8	3
16	4

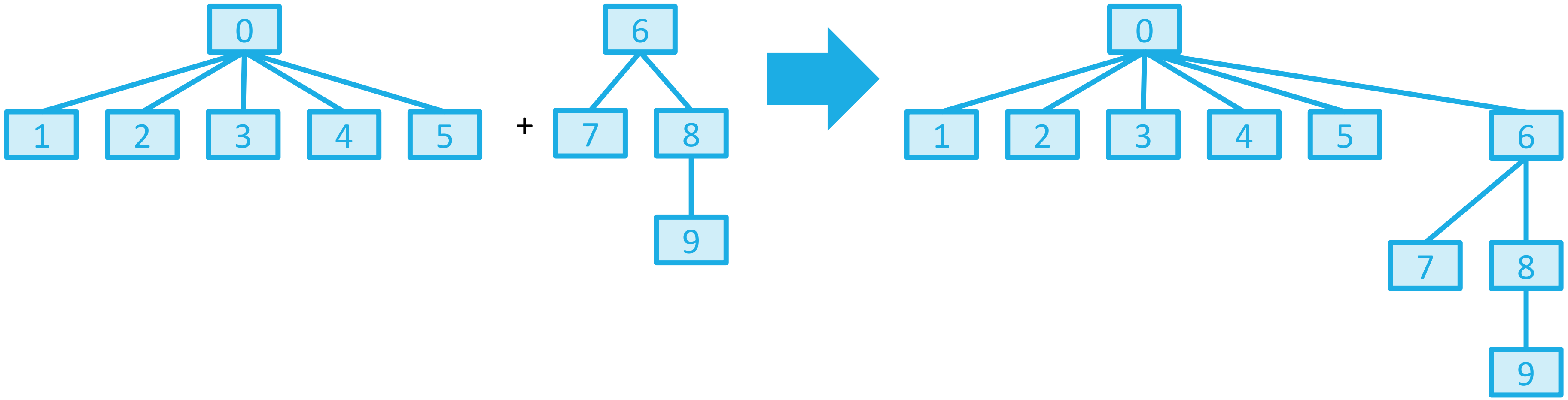


Why Weights Instead of Heights?

We used the number of items in a tree to decide upon the root

Why not use the height of the tree?

- HeightedQuickUnion's runtime is asymptotically the same: Θ(log(n))
- It's easier to track weights than heights, even though WeightedQuickUnion can lead to some suboptimal structures like this one:



WeightedQuickUnion Runtime

	Maps to Sets	QuickFind	QuickUnion	WeightedQuickUnion
`makeSet(value)`	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
`find(value)`	$\Theta(n)$	$\Theta(1)$	$\Theta(n)$	$\Theta(\log n)$
`union(x, y)` `assuming root args`	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$	$\Theta(1)$
`union(x, y)`	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(\log n)$

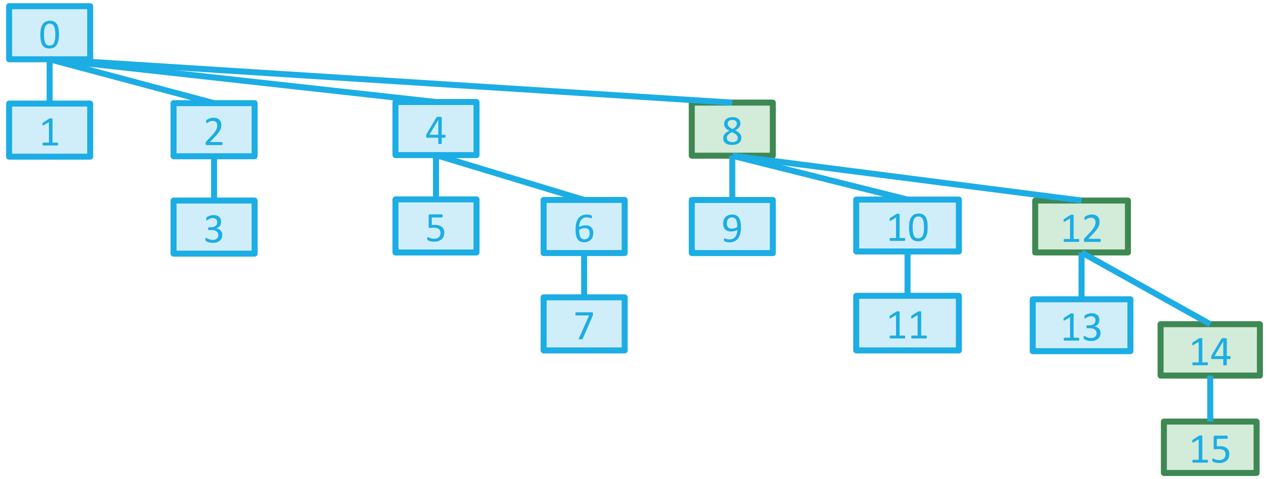
This is pretty good! But there's one final optimization we can make: **path compression**

Modifying Data Structures for Future Gains

- Thus far, the modifications we've studied are designed to *preserve invariants*
  - E.g. Performing rotations to preserve the AVL invariant
  - We rely on those invariants always being true so every call is fast
- Path compression is entirely different: we are modifying the tree structure to *improve future performance*
  - Not adhering to a specific invariant
  - The first call may be slow, but will optimize so future calls can be fast

Path Compression: Idea

This is the worst-case topology if we use WeightedQuickUnion

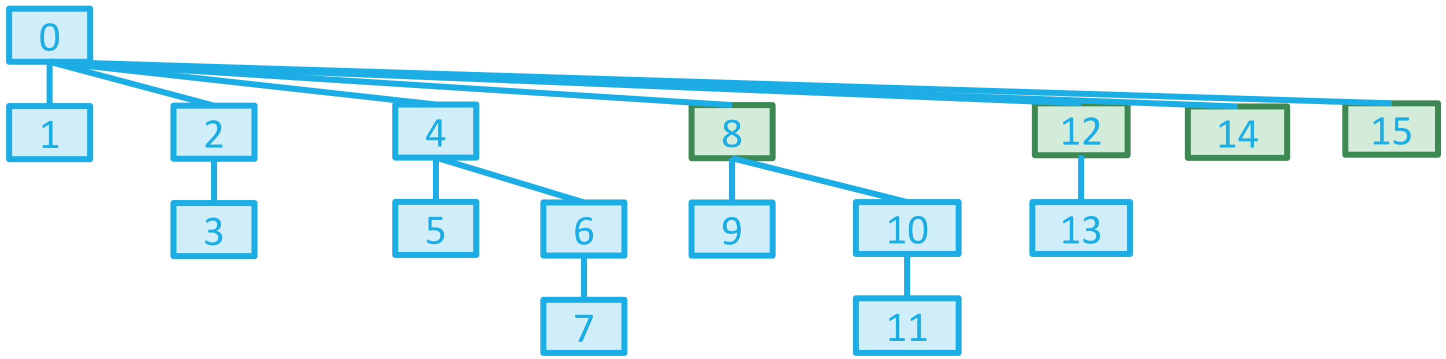


Idea: When we do find(15), move all **visited nodes** under the root

- Additional cost is insignificant (we already have to visit those nodes, just constant time work to point to root too)

Path Compression: Idea

This is the worst-case topology if we use WeightedQuickUnion



Idea: When we do find(15), move all **visited nodes** under the root

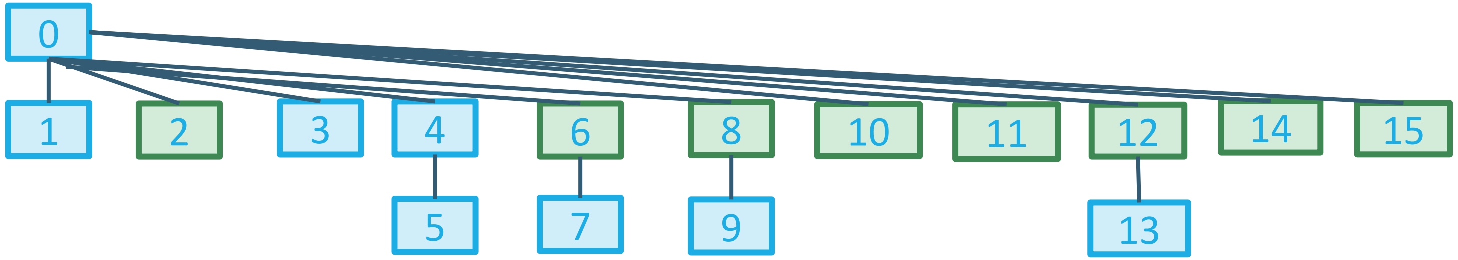
- Additional cost is insignificant (we already have to visit those nodes, just constant time work to point to root too)

Perform Path Compression on every find(), so future calls to find() are faster!

Path Compression: Details and Runtime

Run path compression on every find()!

- Including the find()s that are invoked as part of a union()

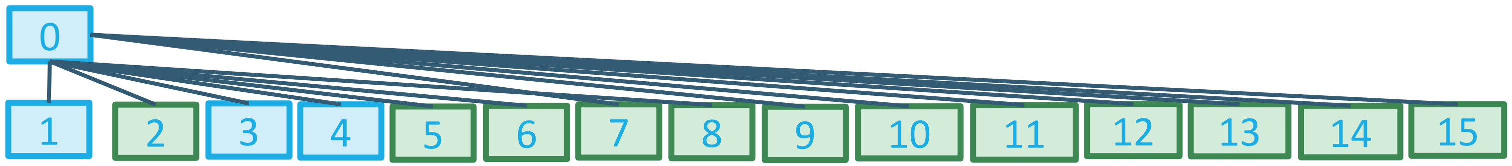


Understanding the performance of M>1 operations requires **amortized analysis**

- Effectively averaging out rare events over many common ones
- Typically used for "In-Practice" case
  - E.g. when we assume an array doesn't resize "in practice", we can do that because the rare resizing calls are *amortized* over many faster calls
- In 373 we don't go in-depth on amortized analysis

Path Compression: Runtime

M find()s on WeightedQuickUnion requires takes Θ(M log N)



… but M find()s on WeightedQuickUnionWithPathCompression takes O(M log*N)!

- log*n is the "iterated log": the number of times you need to apply log to n before it's <= 1
- Note: log* is a loose bound

Path Compression: Runtime

Path compression results in find()s and union()s that are very very close to (amortized) constant time

- log* is less than 5 for any realistic input
- If M find()s/union()s on N nodes is O(M log*N) and log*N ≈ 5, then find()/union()s amortizes to O(1)! 🤯

N	log* N
1	0
2	1
4	2
16	3
65536	4
2^{65536}	5

2^{16}

Number of atoms in the known universe is 2^{256}ish