

Section 03: Recurrences, Master Theorem, Tree Method

Main Problems

1. Finding Bounds

For each of the following code blocks, what is the worst-case runtime? Give a big- Θ bound.

(a)

```
int result = 0;
for (int i = 0; i < n; i++) {
    for (int j = 0; j < i; j++) {
        result++;
    }
}
```

(b)

```
public IList<String> repeat(DoubleLinkedList<String> list, int n) {
    IList<String> result = new DoubleLinkedList<String>();
    for(String str : list) {
        for(int i = 0; i < n; i++) {
            result.add(str);
        }
    }
    return result;
}
```

(c)

```
public void foo(int n) {
    for (int i = 0; i < n; i++) {
        for (int j = 5; j < i; j++) {
            System.out.println("Hello!");
        }

        for (int j = i; j >= 0; j -= 2) {
            System.out.println("Hello!");
        }
    }
}
```

(d)

```
public int num(int n){
    if (n < 10) {
        return n;
    } else if (n < 1000) {
        return num(n - 2);
    } else {
        return num(n / 2);
    }
}
```

```
(e) public int foo(int n) {
    if (n <= 0) {
        return 3;
    }
    int x = foo(n - 1);
    System.out.println("hello");
    x += foo(n - 1);
    return x;
}
```

2. Binary Search Trees

(a) Write a method `validate` to validate a BST. Although the basic algorithm can be converted to any data structure and work in any format, if it helps, you may write this method for the `IntTree` class:

```
public class IntTree {
    private IntTreeNode overallRoot;

    // constructors and other methods omitted for clarity

    private class IntTreeNode {
        public int data;
        public IntTreeNode left;
        public IntTreeNode right;

        // constructors omitted for clarity
    }
}
```

3. Code To Recurrence

(a) Consider the following method.

```
public static int f(int N) {
    if (N <= 1) {
        return 0;
    }

    int result = 0;
    for (int i = 0; i < N; i++) {
        for (int j = 0; j < i; j++) {
            result++;
        }
    }

    return 5 * f(N / 2) + 3 * result + 2 * f(N / 2) + f(N / 2) + f(N / 2);
}
```

Give a recurrence formula for the running time of this code. It's OK to provide a \mathcal{O} for the non-recursive terms (for example if the running time is $A(n) = 4A(n/3) + 25n$, you need to get the 4 and the 3 right but you don't have to worry about getting the 25 right). Just show us how you got there.

Hint: Notice that the main loop is the exact same code as Problem 1A.

(b) Consider the following method.

```
public static int g(n) {
    if (n <= 1) {
        return 1000;
    }
    if (g(n / 3) > 5) {
        for (int i = 0; i < n; i++) {
            System.out.println("Hello");
        }
        return 5 * g(n / 3);
    } else {
        for (int i = 0; i < n * n; i++) {
            System.out.println("World");
        }
        return 4 * g(n / 3);
    }
}
```

(i) Find a recurrence $S(n)$ modeling the worst-case runtime of $g(n)$.

(ii) Find a recurrence $X(n)$ modeling the *returned integer output* of $g(n)$.

(iii) Find a recurrence $P(n)$ modeling the *printed output* of $g(n)$.

(c) Consider the following set of recursive methods.

```
public int test(int n) {
    IDictionary<Integer, Integer> dict = new AvlDictionary<>();
    populate(n, dict);
    int counter = 0;
    for (int i = 0; i < n; i++) {
        counter += dict.get(i);
    }
    return counter;
}

private void populate(int k, IDictionary<Integer, Integer> dict) {
    if (k == 0) {
        dict.put(0, k);
    } else {
        for (int i = 0; i < k; i++) {
            dict.put(i, i);
        }
        populate(k / 2, dict);
    }
}
```

(i) Write a mathematical function representing the *worst-case runtime* of `test`.

You should write two functions, one for the runtime of `test` and one for the runtime of `populate`.

4. Master Theorem

For each of the recurrences below, use the Master Theorem to find the big- Θ of the closed form or explain why Master Theorem doesn't apply. (See the last page for the definition of Master Theorem.)

$$(a) T(n) = \begin{cases} 18 & \text{if } n \leq 5 \\ 3T(n/4) + n^2 & \text{otherwise} \end{cases}$$

$$(b) T(n) = \begin{cases} 1 & \text{if } n \leq 1 \\ 9T(n/3) + n^2 & \text{otherwise} \end{cases}$$

$$(c) T(n) = \begin{cases} 1 & \text{if } n \leq 1 \\ \log(n)T(n/2) + n & \text{otherwise} \end{cases}$$

$$(d) T(n) = \begin{cases} 1 & \text{if } n \leq 19 \\ 4T(n/3) + n & \text{otherwise} \end{cases}$$

$$(e) T(n) = \begin{cases} 5 & \text{if } n \leq 24 \\ 2T(n-2) + 5n^3 & \text{otherwise} \end{cases}$$

5. Tree method walk-through

Consider the following recurrence: $A(n) = \begin{cases} 1 & \text{if } n \leq 1 \\ 3A(n/6) + n & \text{otherwise} \end{cases}$

We want to find an *exact* closed form of this equation by using the tree method. Suppose we draw out the total work done by this method as a tree, as discussed in lecture. Let n be the initial input to A .

(a) What is the size of the input at level i (as in class, call the root level 0)?

(b) What is the work done by a single node at non-base-case level i ?

(c) What is the number of nodes at level i ?

Note: let $i = 0$ indicate the level corresponding to the root node. So, when $i = 0$, your expression should be equal to 1.

(d) What is the total work at the i^{th} **recursive** level?

(e) What is the last level of the tree?

(f) What is the work done in the base case?

(g) Combine your answers from previous parts to get an expression for the total work.

(h) Simplify to a closed form.

Note: you do not need to simplify your answer, once you found the closed form. Hint: You should use the finite geometric series identity somewhere while finding a closed form.

(i) Use the master theorem to find a big- Θ bound of $A(n)$. (See the last page for the definition of Master Theorem.)

6. More tree method recurrences

For each of the following recurrences, find their closed form using the tree method. Then, check your answer using the master method (if applicable). It may be a useful guide to use the steps from section 4 of this handout to help you with all the parts of solving a recurrence problem fully.

$$(a) T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 3 & \text{otherwise} \end{cases}$$

$$(b) S(q) = \begin{cases} 1 & \text{if } q = 1 \\ 2S(q-1) + 1 & \text{otherwise} \end{cases}$$

$$(c) T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 8T(n/2) + 4n^2 & \text{otherwise} \end{cases}$$

Challenge Problems

7. TreeMap implemented as a Binary Search Tree

Consider the following method, which is a part of a Binary Search Tree implementation of a TreeMap class.

```
public V find(K key) {
    return find(this.root, key);
}

private V find(Node<K, V> current, K key) {
    if (current == null) {
        return null;
    }
    if (current.key.equals(key)) {
        return current.value;
    }
    if (current.key.compareTo(key) > 0) {
        return find(current.left, key);
    } else {
        return find(current.right, key);
    }
}
```

- (a) We want to analyze the runtime of our `find(x)` method in the best possible case and the worst possible case. What does our tree look like in the best possible case? In the worst possible case?
- (b) Write a recurrence to represent the worst-case runtime for `find(x)` in terms of n , the number of elements contained within our tree. Then, provide a Θ bound.
- (c) Assuming we have an optimally structured tree, write a recurrence for the runtime of `find(x)` (again in terms of n). Then, provide a Θ bound.

8. Maximum Difference Between Node and Ancestor

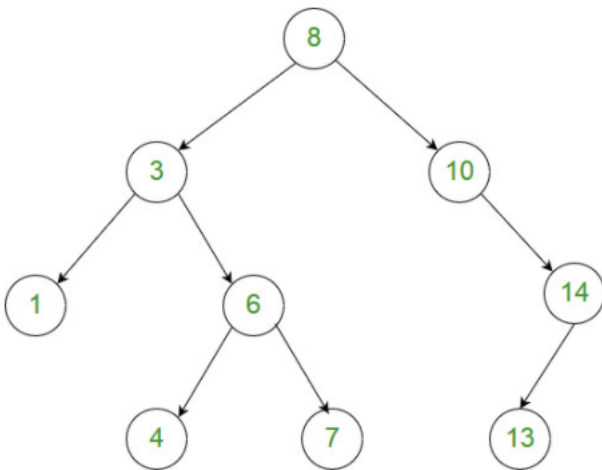
- (a) Let a `TreeNode` be defined as the following:

```
public class TreeNode {
    int val;
    TreeNode left;
    TreeNode right;
    TreeNode() {}
    TreeNode(int val) { this.val = val; }
    TreeNode(int val, TreeNode left, TreeNode right) {
        this.val = val;
        this.left = left;
        this.right = right;
    }
}
```

Given the root of a binary tree, find the maximum value V for which there exist different nodes A and B where $V = |A.val - B.val|$ and A is an ancestor of B .

A node A is an ancestor of B if either any child of A is equal to B , **or** any child of A is an ancestor of B .

Consider the following example:



Input: [8, 3, 10, 1, 6, null, 14, null, null, 4, 7, 13]

Output: 7

Explanation: We have various ancestor-node differences, some of which are given below:

$$|8 - 3| = 5$$

$$|3 - 7| = 4$$

$$|8 - 1| = 7$$

$$|10 - 13| = 3$$

Among all possible differences, the maximum value of 7 is obtained by $|8 - 1| = 7$.

Master Theorem

For recurrences in this form, where a, b, c, e are constants:

$$T(n) = \begin{cases} d & \text{if } n \leq \text{some constant} \\ aT(n/b) + e \cdot n^c & \text{otherwise} \end{cases} \quad T(n) \text{ is } \begin{cases} \Theta(n^c) & \text{if } \log_b(a) < c \\ \Theta(n^c \log n) & \text{if } \log_b(a) = c \\ \Theta(n^{\log_b(a)}) & \text{if } \log_b(a) > c \end{cases}$$

Useful summation identities

Splitting a sum

$$\sum_{i=a}^b (x + y) = \sum_{i=a}^b x + \sum_{i=a}^b y$$

Adjusting summation bounds

$$\sum_{i=a}^b f(x) = \sum_{i=0}^b f(x) - \sum_{i=0}^{a-1} f(x)$$

Factoring out a constant

$$\sum_{i=a}^b cf(i) = c \sum_{i=a}^b f(i)$$

Summation of a constant

$$\sum_{i=0}^{n-1} c = \underbrace{c + c + \dots + c}_{n \text{ times}} = cn$$

Note: this rule is a special case of the rule on the left

Gauss's identity

$$\sum_{i=0}^{n-1} i = 0 + 1 + 2 + \dots + n - 1 = \frac{n(n-1)}{2}$$

Finite geometric series

$$\sum_{i=0}^{n-1} x^i = 1 + x + x^2 + \dots + x^{n-1} = \frac{x^n - 1}{x - 1}$$

Infinite geometric series

$$\sum_{i=0}^{\infty} x^i = 1 + x + x^2 + \dots = \frac{1}{1-x}$$

Note: applicable only when $-1 < x < 1$

Useful Log Rules

Power of a log identity

$$a^{\log_b c} = c^{\log_b a}$$

Product rule

$$\log_c(a * b) = \log_c a + \log_c b$$

Quotient rule

$$\log_c(a/b) = \log_c a - \log_c b$$

Power rule

$$\log_c(a^b) = b * \log_c a$$

Change of base formula

$$\log_b a = (\log_c a) / (\log_c b)$$