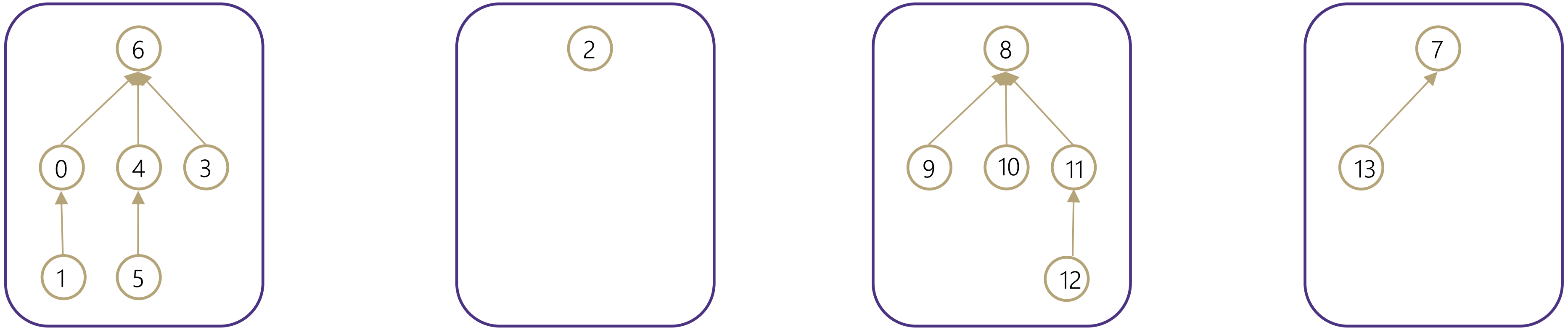
Lecture 19: Disjoint Sets

CSE 373: Data Structures and Algorithms

Practice

Given the following disjoint-set what would be the result of the following calls on union if we add the “union-by-weight” optimization. Draw the forest at each stage with corresponding ranks for each tree.




```
union(2, 13)
union(4, 12)
union(2, 8)
```

Practice

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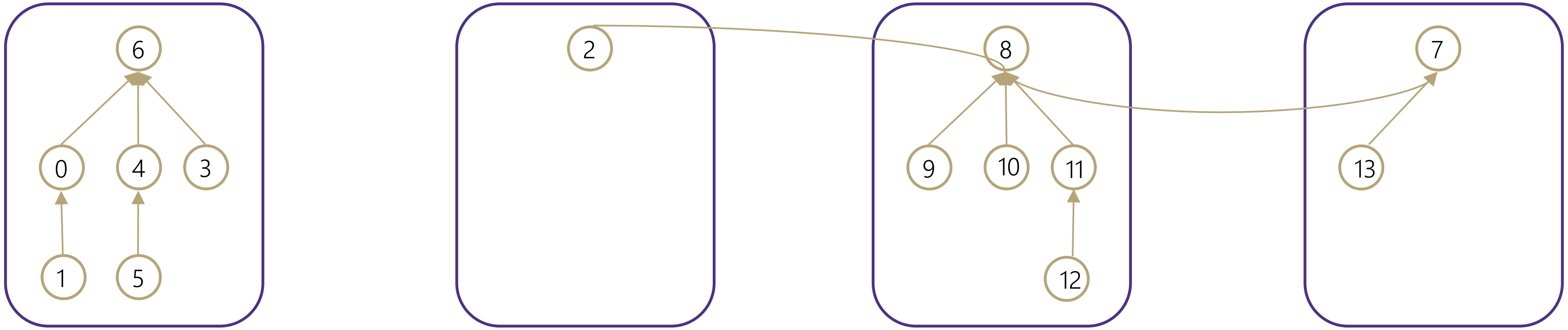


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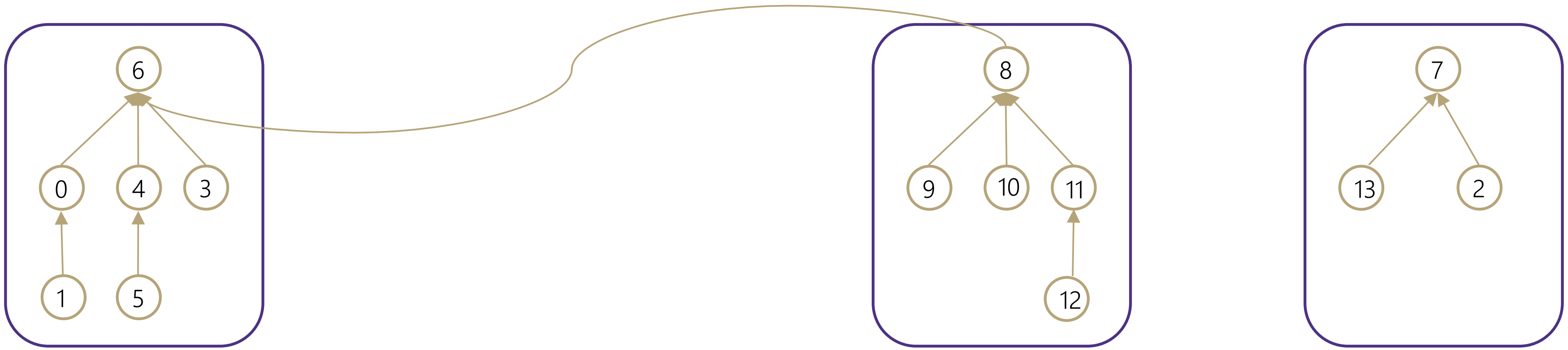


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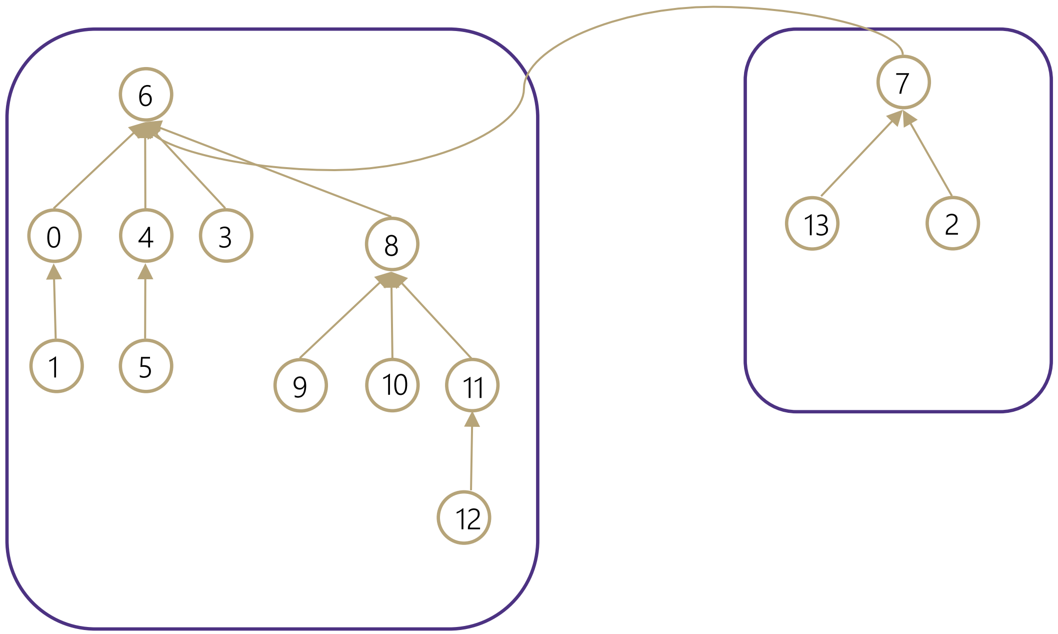


```
union(2, 13)
union(4, 12)
union(2, 8)
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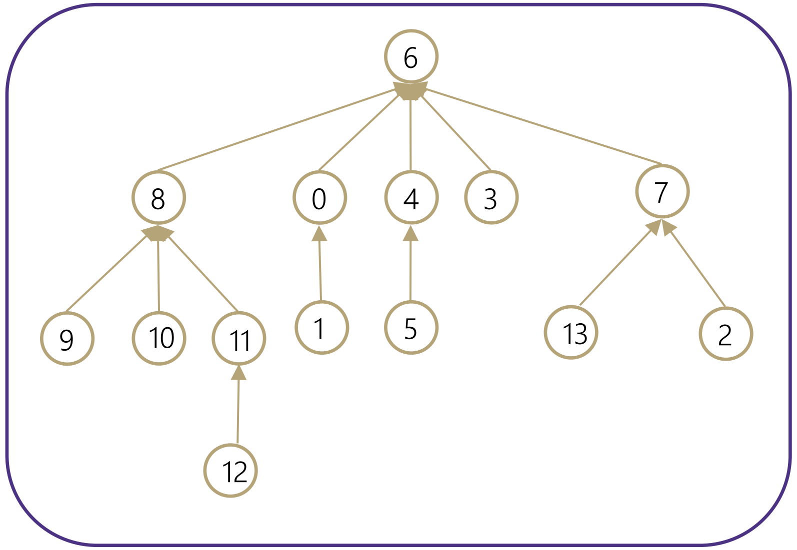
Practice

Given the following disjoint-set what would be the result of the following calls on union if we add the "union-by-weight" optimization. Draw the forest at each stage with corresponding ranks for each tree.



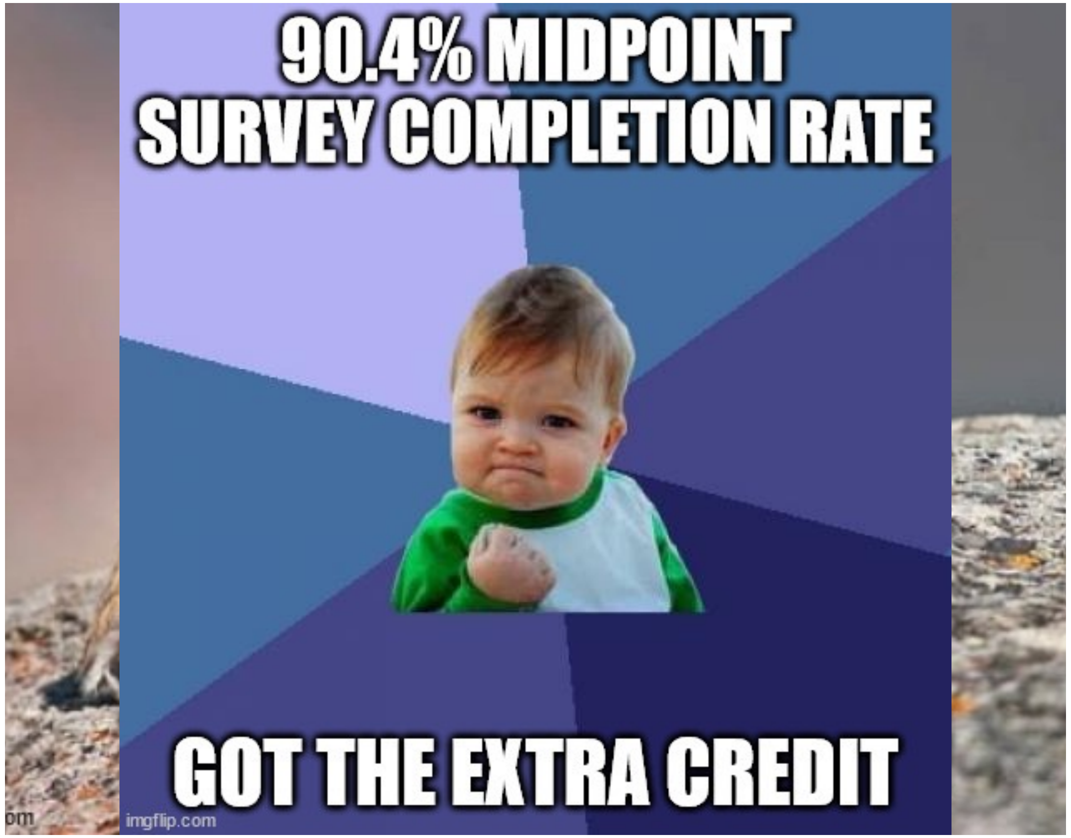

```
union(2, 13)
union(4, 12)
union(2, 8)
```

Does this improve the worst case runtimes?

findSet is more likely to be O(log(n)) than O(n)

Midpoint survey

Thank you all so much for filling out the lecture and section midpoint surveys! We appreciate the feedback and are working on incorporating it :)



Announcements

P3 due Today

P4 comes out today due in 3 weeks on Wednesday June 2nd
- last project!
- ~2 weeks of work
- extra credit spec quiz on gradescope!

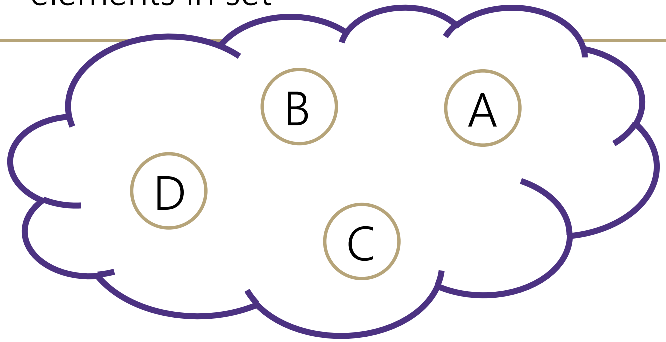
E3 came out on Friday – due this Friday May 14th
- two more exercises coming

Upcoming meme competition

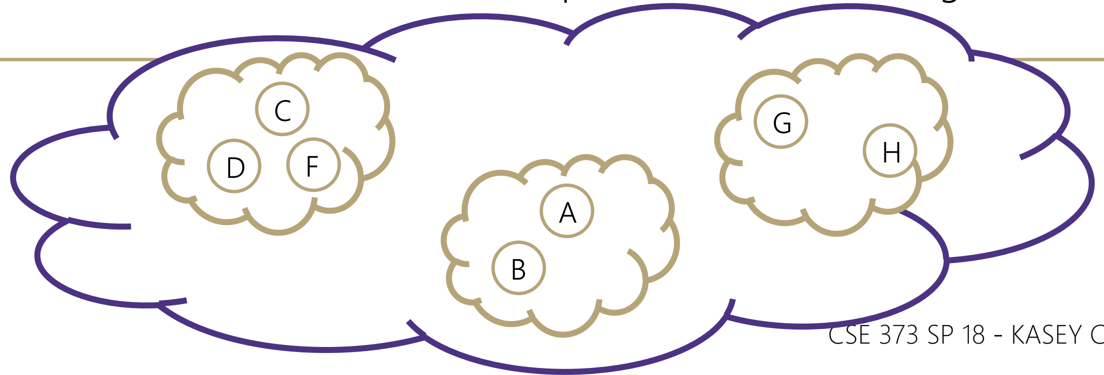
Reminders:
- Tons of extra practice on section hand outs
- section slides and videos are also available
- always always feel free to reach out to Tas ☺

New ADT

Set ADT
state Set of elements - Elements must be unique! - No required order Count of Elements **behavior** create(x) - creates a new set with a single member, x add(x) - adds x into set if it is unique, otherwise add is ignored remove(x) – removes x from set size() – returns current number of elements in set



Disjoint-Set ADT
state Set of Sets - Disjoint: Elements must be unique across sets - No required order - Each set has representative Count of Sets **behavior** makeSet(x) – creates a new set within the disjoint set where the only member is x. Picks representative for set findSet(x) – looks up the set containing element x, returns representative of that set union(x, y) – looks up set containing x and set containing y, combines two sets into one. Picks new representative for resulting set



Implementation

Disjoint-Set ADT

state

Set of Sets
- **Disjoint:** Elements must be unique across sets
- No required order
- Each set has representative

Count of Sets

behavior

makeSet(x) – creates a new set within the disjoint set where the only member is x. Picks representative for set

findSet(x) – looks up the set containing element x, returns representative of that set

union(x, y) – looks up set containing x and set containing y, combines two sets into one. Picks new representative for resulting set

TreeDisjointSet<E>

state

```
Collection<TreeSet> forest
Dictionary<NodeValues, NodeLocations> nodeInventory
```

behavior

```
makeSet(x)-create a new tree of size 1 and add to our forest

findSet(x)-locates node with x and moves up tree to find root

union(x, y)-append tree with y as a child of tree with x
```

TreeSet<E>

state

```
SetNode overallRoot
```

behavior

```
TreeSet(x)
add(x)
remove(x, y)
getRep()-returns data of overallRoot
```

SetNode<E>

state

```
E data
Collection<SetNode> children
```

behavior

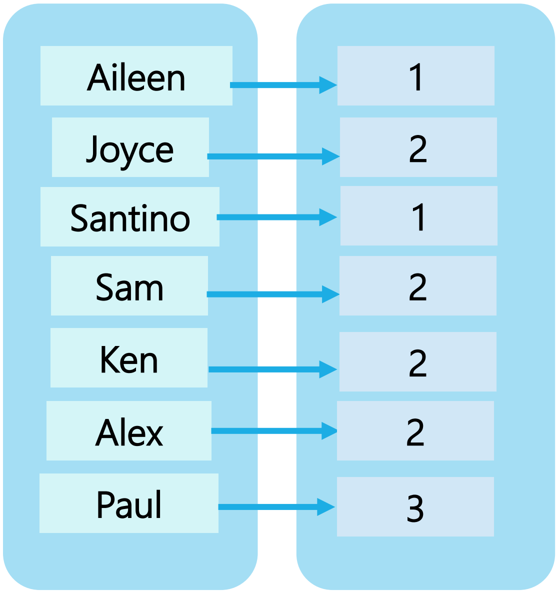
```
SetNode(x)
addChild(x)
removeChild(x, y)
```

Review QuickFind vs. QuickUnion

DISJOINT SETS ADT

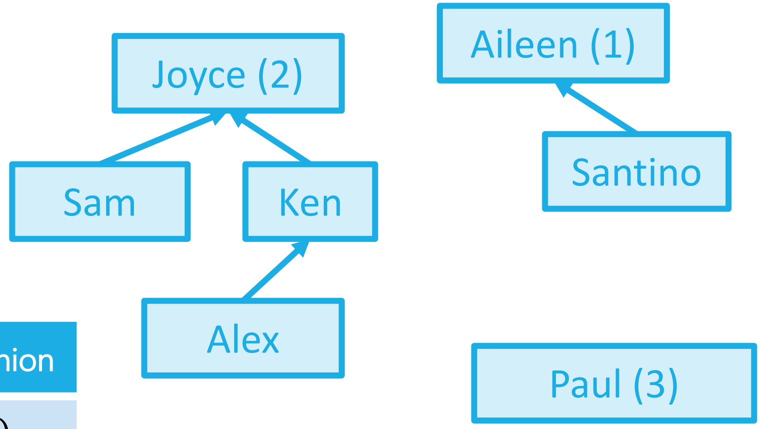
QuickFind

map from value to representative ID



QuickUnion

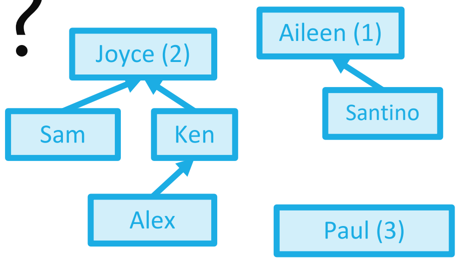
trees of values with representative ID at each root



Could also use one element from each set (e.g. the root) as its representative: only uniqueness matters

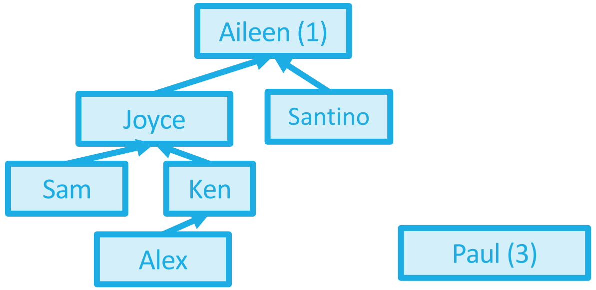
	(Baseline)	QuickFind	QuickUnion
`makeSet(value)`	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
`find(value)`	$\Theta(n)$	$\Theta(1)$	$\Theta(n)$
`union(x, y)`	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$

Review QuickUnion: Why Use Both Roots?



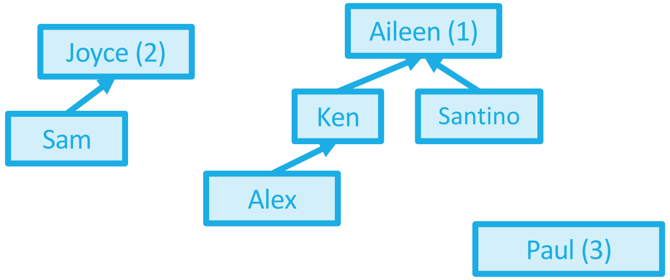
Example: result of `union(Ken, Santino)` on these Disjoint Sets given three possible implementations:

```
union(A, B):
  rootA = find(A)
  rootB = find(B)
  set rootA to point to rootB
```



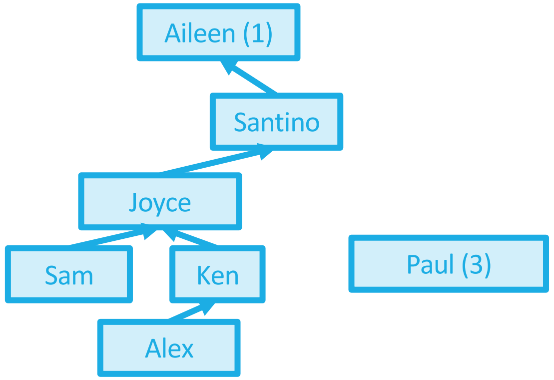
Correct: Everything in Ken's set now connected to everything in Santino's set!

```
union(A, B):
  rootB = find(B)
  set A to point to rootB
```



Incorrect: Ken and Joyce were connected before; the union operation shouldn't remove connections.

```
union(A, B):
  rootA = find(A)
  set rootA to point to B
```



Inefficient: Technically correct, but increases height of the up-tree so makes

Review WeightedQuickUnion

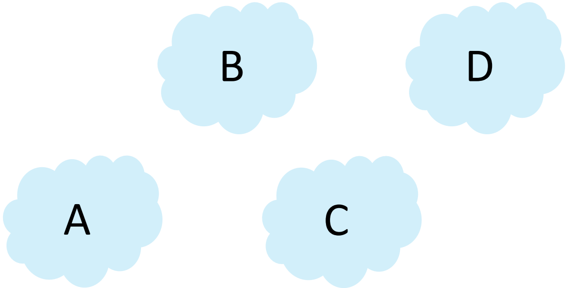
Goal: Always pick the smaller tree to go under the larger tree

Implementation: Store the number of nodes (or “weight”) of each tree in the root

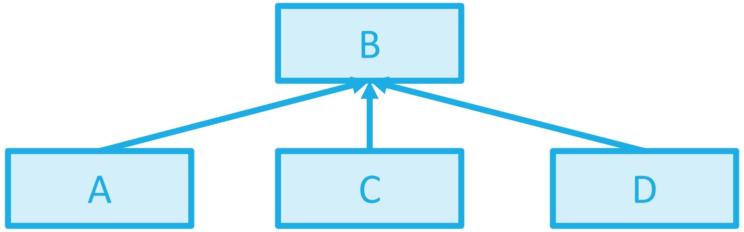
- Constant-time lookup instead of having to traverse the entire tree to count

```
union(A, B):
  rootA = find(A)
  rootB = find(B)
  put lighter root under heavier root
```

```
union(A, B)
union(B, C)
union(C, D)
find(A)
```



Now what happens?



Perfect! Best runtime we can get.

Review WeightedQuickUnion: Performance

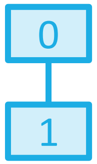
Consider the worst case where the tree height grows as fast as possible

N	H
1	0

Review WeightedQuickUnion: Performance

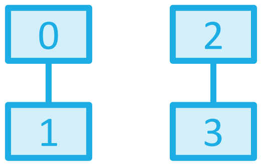
Consider the worst case where the tree height grows as fast as possible

N	H
1	0
2	1



Review WeightedQuickUnion: Performance

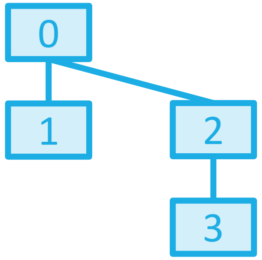
Consider the worst case where the tree height grows as fast as possible



N	H
1	0
2	1
4	?

Review WeightedQuickUnion: Performance

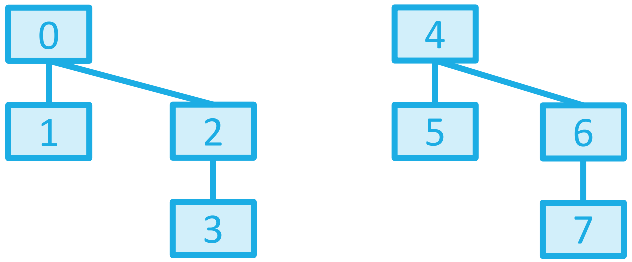
Consider the worst case where the tree height grows as fast as possible



N	H
1	0
2	1
4	2

Review WeightedQuickUnion: Performance

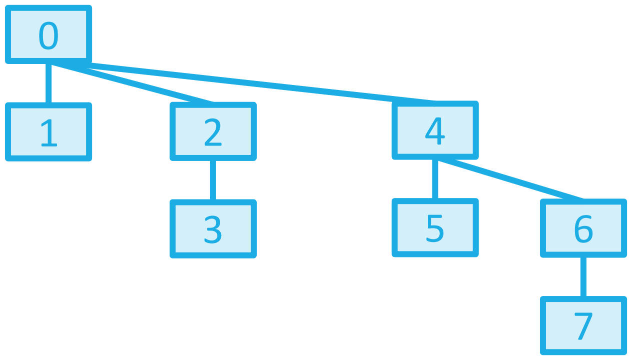
Consider the worst case where the tree height grows as fast as possible



N	H
1	0
2	1
4	2
8	?

Review WeightedQuickUnion: Performance

Consider the worst case where the tree height grows as fast as possible

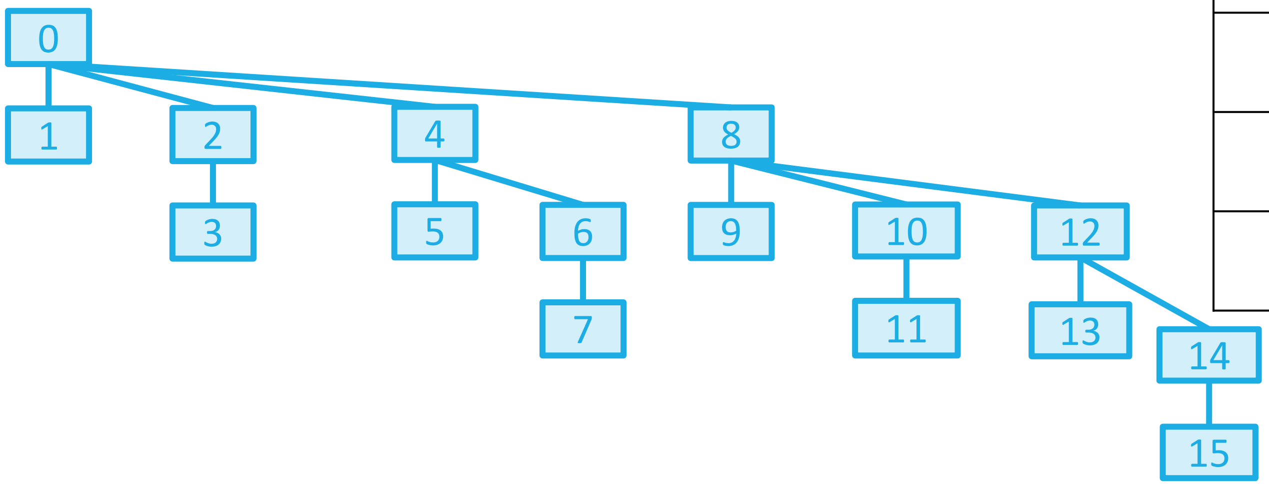


N	H
1	0
2	1
4	2
8	3

Review WeightedQuickUnion: Performance

- Consider the worst case where the tree height grows as fast as possible
- Worst case tree height is Θ(log N)

N	H
1	0
2	1
4	2
8	3
16	4

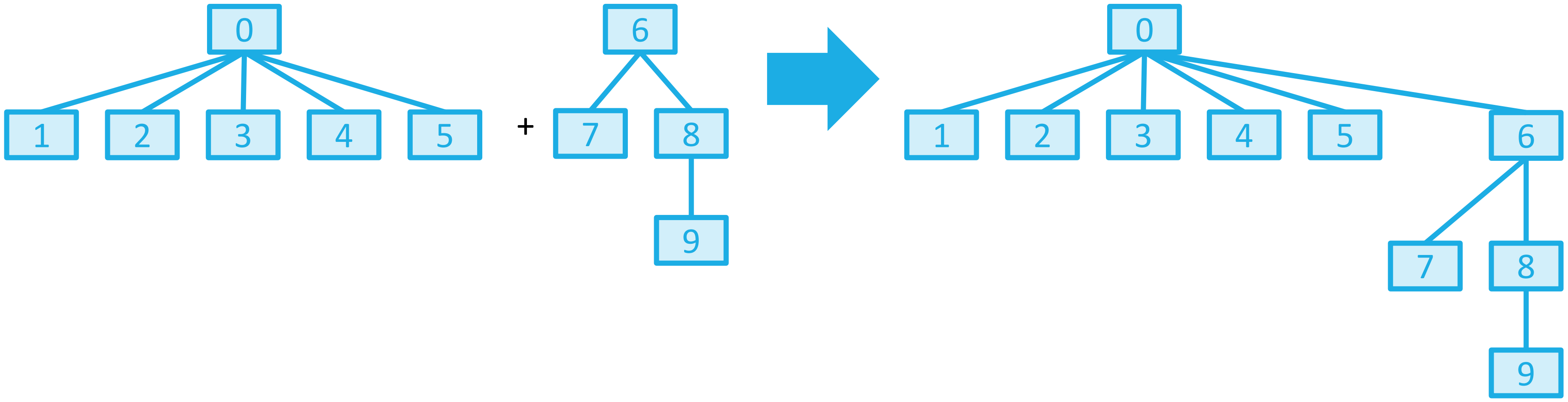


Review Why Weights Instead of Heights?

We used the number of items in a tree to decide upon the root

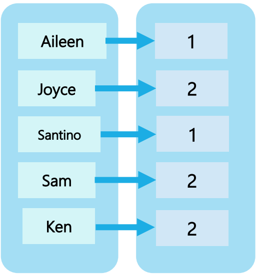
Why not use the height of the tree?

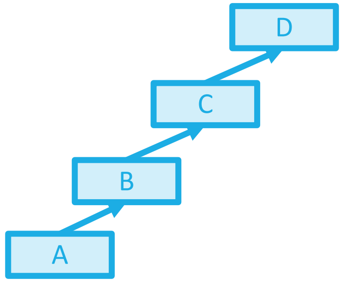
- HeightedQuickUnion's runtime is asymptotically the same: Θ(log(n))
- It's easier to track weights than heights, even though WeightedQuickUnion can lead to some suboptimal structures like this one:

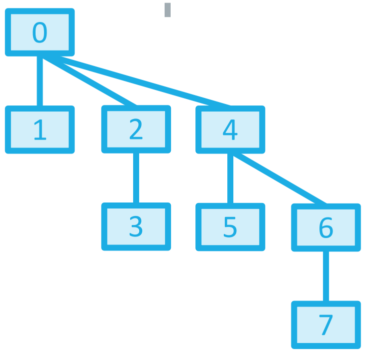


Review WeightedQuickUnion Runtime

	(Baseline)	QuickFind	QuickUnion	WeightedQuickUnion
`makeSet(value)`	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
`find(value)`	$\Theta(n)$	$\Theta(1)$	$\Theta(n)$	$\Theta(\log n)$
`union(x, y)` `assuming root args`	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$	$\Theta(1)$
`union(x, y)`	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(\log n)$







This is pretty good! But there’s one final optimization we can make: **path compression**

Modifying Data Structures for Future Gains

Thus far, the modifications we've studied are designed to *preserve invariants*

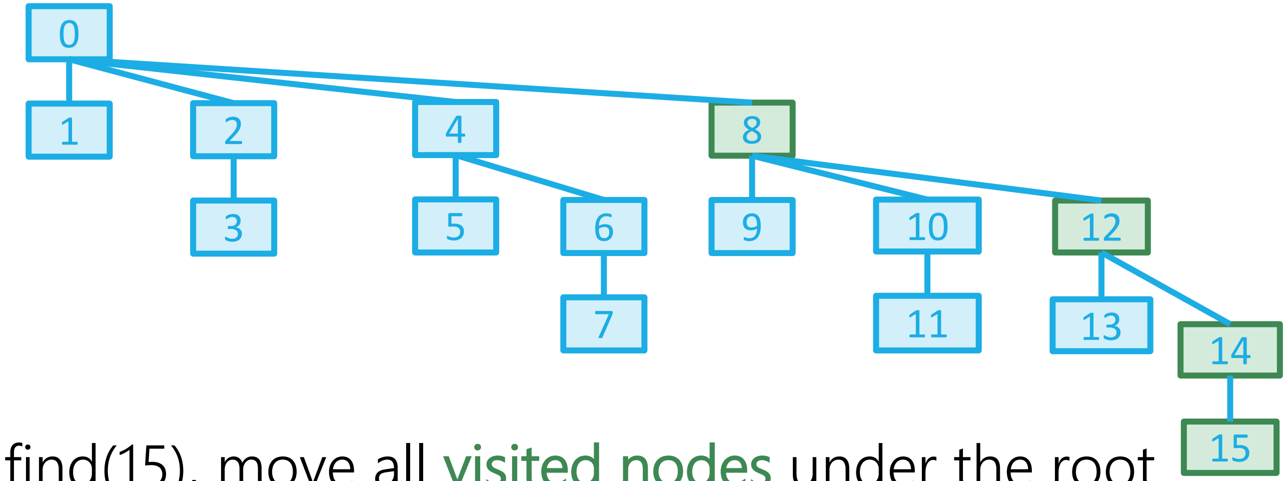
- E.g. Performing rotations to preserve the AVL invariant
- We rely on those invariants always being true so every call is fast

Path compression is entirely different: we are modifying the tree structure to *improve future performance*

- Not adhering to a specific invariant
- The first call may be slow, but will optimize so future calls can be fast

Path Compression: Idea

This is the worst-case topology if we use WeightedQuickUnion



Idea: When we do find(15), move all visited nodes under the root

- Additional cost is insignificant (we already have to visit those nodes, just constant time work to point to root too)

Path Compression: Idea

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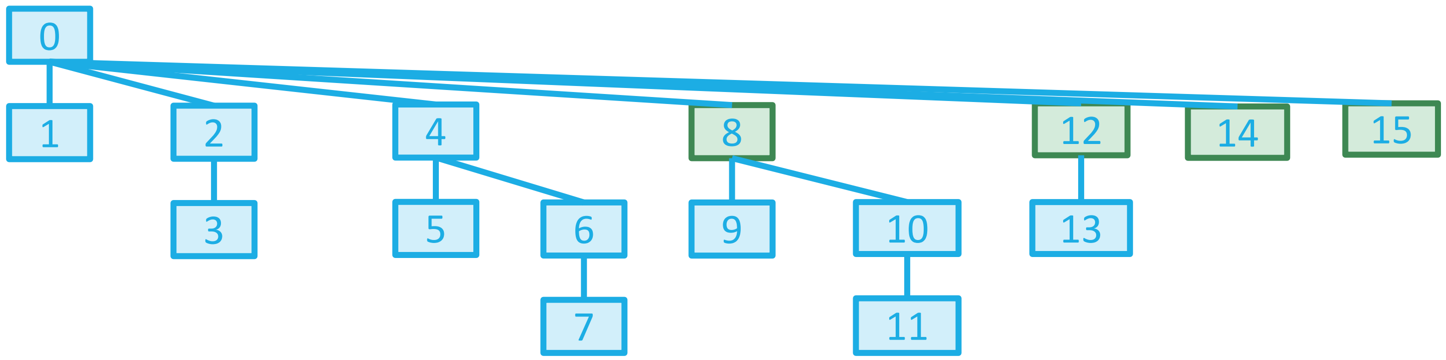


Idea: When we do find(15), move all visited nodes under the root

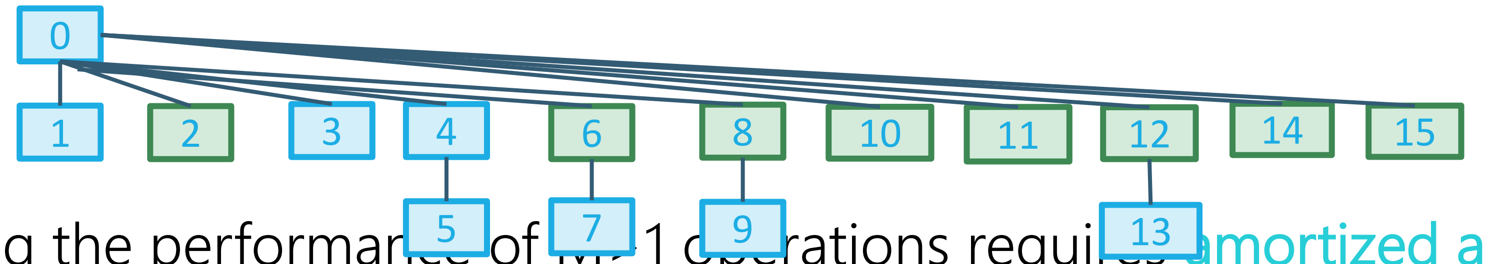
- Additional cost is insignificant (we already have to visit those nodes, just constant time work to point to root too)

- Perform Path Compression on every find(), so future calls to find() are faster!

Path Compression: Details and Runtime

Run path compression on every find()!

- Including the find()s that are invoked as part of a union()

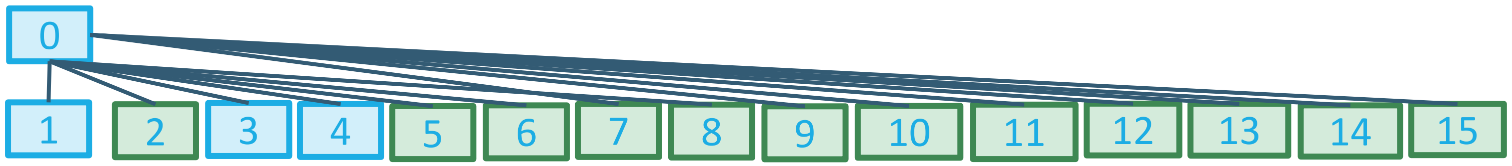


Understanding the performance of $M>1$ operations requires amortized analysis

- Effectively averaging out rare events over many common ones
- Typically used for “In-Practice” case
  - E.g. when we assume an array doesn’t resize “in practice”, we can do that because the rare resizing calls are *amortized* over many faster calls
- In 373 we don’t go in-depth on amortized analysis

Path Compression: Runtime

M find()s on WeightedQuickUnion requires takes Θ(M log N)



... but M find()s on WeightedQuickUnionWithPathCompression takes O(M log*N)!

- log*n is the "iterated log": the number of times you need to apply log to n before it's <= 1
- Note: log* is a loose bound

Path Compression: Runtime

Path compression results in find()s and union()s that are very very close to (amortized) constant time

- log* is less than 5 for any realistic input
- If M find()s/union()s on N nodes is O(M log*N) and log*N ≈ 5, then find()/union()s amortizes to O(1)! 🤯

N	log* N
1	0
2	1
4	2
16	3
65536	4
2^{65536}	5

2^{16}

Number of atoms in the known universe is 2^{256}ish

WQU + Path Compression Runtime

In-Practice Runtimes:

	(Baseline)	QuickFind	QuickUnion	WeightedQuickUnion	WQU + Path Compression
`makeSet(value)`	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
`find(value)`	$\Theta(n)$	$\Theta(1)$	$\Theta(n)$	$\Theta(\log n)$	$O(\log^* n)$
`union(x, y)` `assuming root args`	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
`union(x, y)`	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(\log n)$	$O(\log^* n)$

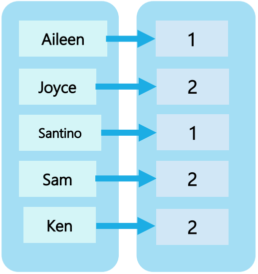
And if log* n <= 5 for any reasonable input…

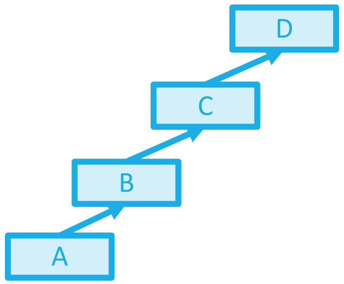
- We've just witnessed an incredible feat of data structure engineering: every operation is constant!?*
- *Caveat: *amortized* constant, in the "in-practice" case; still logarithmic in the worst case!

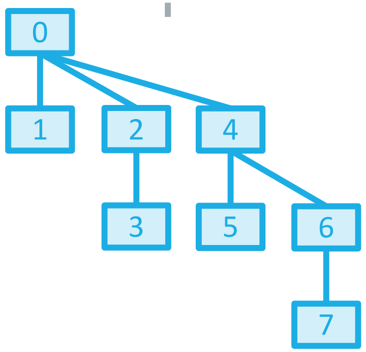
Disjoint Sets Implementation

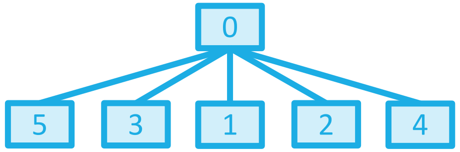
In-Practice Runtimes:

	(Baseline)	QuickFind	QuickUnion	WeightedQuickUnion	WQU + Path Compression
`makeSet(value)`	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
`find(value)`	$\Theta(n)$	$\Theta(1)$	$\Theta(n)$	$\Theta(\log n)$	$O(\log^* n)$
`union(x, y)` `assuming root args`	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
`union(x, y)`	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(\log n)$	$O(\log^* n)$









Kruskal's Runtime

$\Theta(|E| \log |E|)$

$\Theta(|V|)$

$\Theta(|E| \log |E|)$

$\Theta(|E| \log |V|)$: $\Theta(|E|)$ iterations, $\Theta(\log |V|)$

$\Theta(|V| \log |V|)$: total $\Theta(|V|)$ iterations, $\Theta(\log |V|)$

```
kruskalMST(G graph)
  DisjointSets<V> msts; Set finalMST;
  initialize msts with each vertex as single-element MST
  sort all edges by weight (smallest to largest)

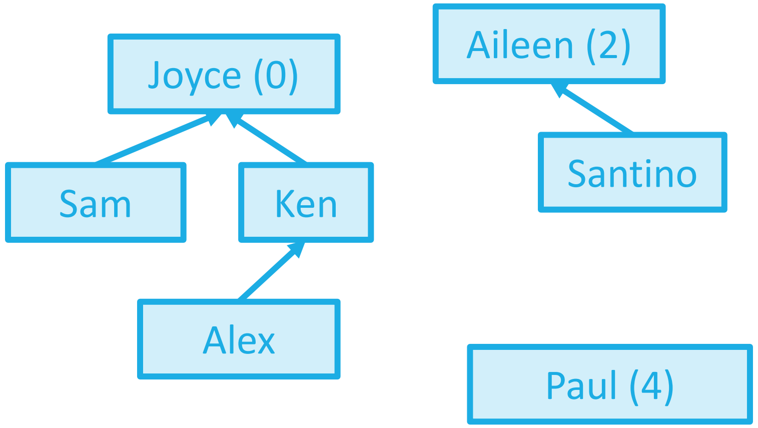
  for each edge (u,v) in ascending order:
    uMST = msts.find(u)
    vMST = msts.find(v)
    if (uMST != vMST):
      finalMST.add(edge (u, v))
      msts.union(uMST, vMST)
```

find and union are log|V| in worst case, but amortized constant "in practice"

Either way, dominated by time to sort the edges ☹

- For an MST to exist, E can't be smaller than V, so assume it dominates
- Note: some people write |E|log|V|, which is the same (within a constant factor)

Using Arrays for Up-Trees



Since every node can have at most one parent, what if we use an array to store the parent relationships?

Proposal: each node corresponds to an index, where we store the index of the parent (or –1 for roots). Use the root index as the representative ID!

Just like with heaps, tree picture still conceptually correct, but exists in our minds!

0	1	2	3	4	5	6
-1	0	-1	6	-1	2	0
Joyce	*Sam*	*Aileen*	*Alex*	*Paul*	*Santino*	*Ken*

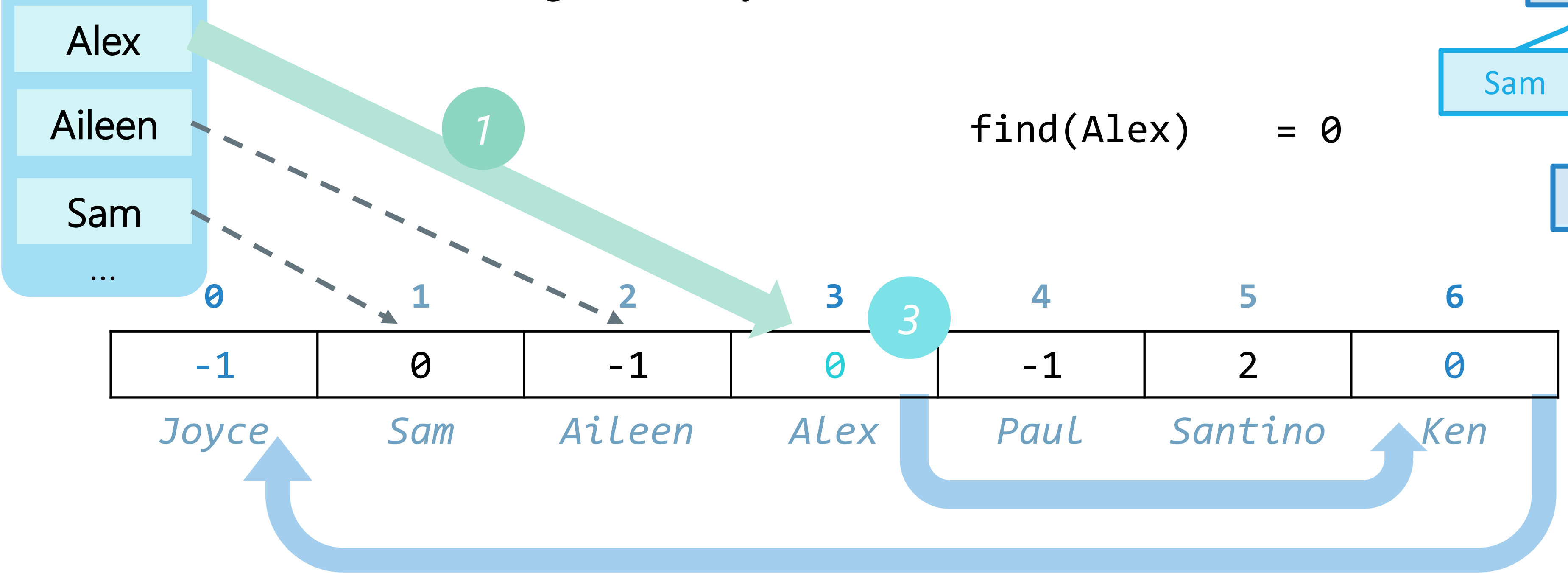
Using Arrays: Find

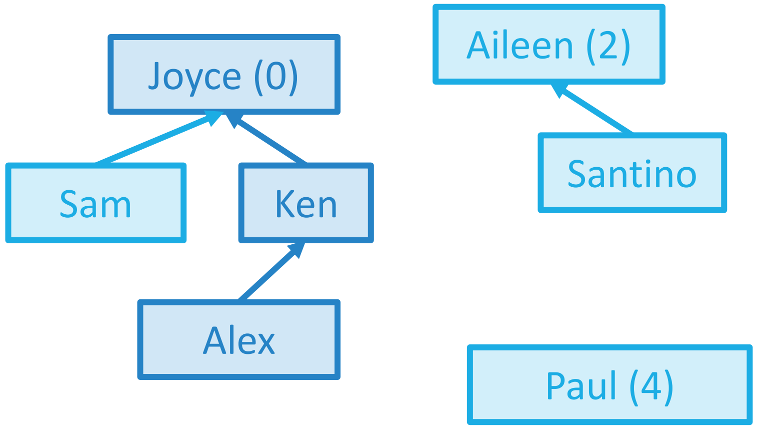
Initial jump to element still done with extra Map

But traversing up the tree can be done purely within the array!

- Can still do path compression by setting all indices along the way to the root index!

```
find(A):
  index = jump to A node's index
  while array[index] > 0:
    index = array[index]
  path compression
  return index
```



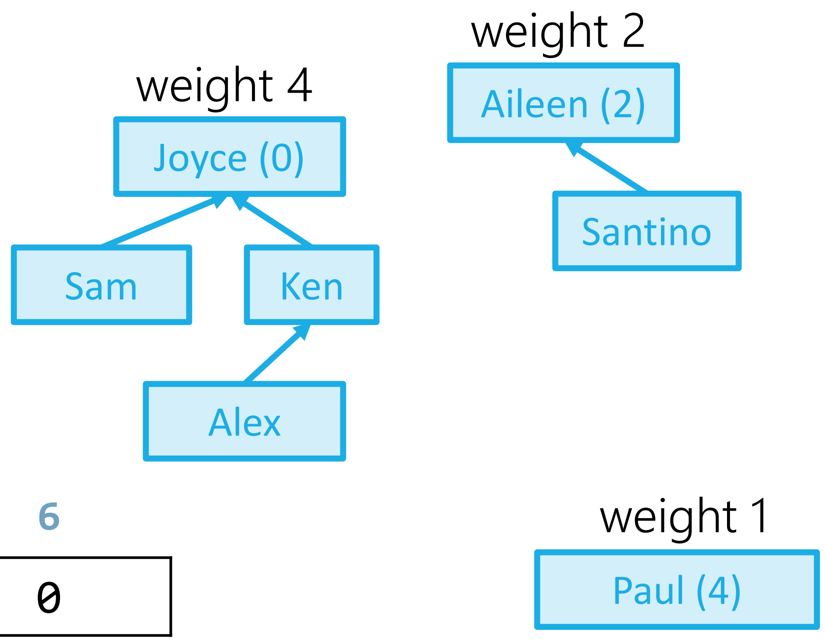


Using Arrays: Union

For WeightedQuickUnion, we need to store the number of nodes in each tree (the weight)

Instead of just storing -1 to indicate a root, we can store -1 * weight!

```
union(A, B):
  rootA = find(A)
  rootB = find(B)
  use -1 * array[rootA] and -1 *
       array[rootB] to determine weights
  put lighter root under heavier root
```



`union(Ken, Santino)`

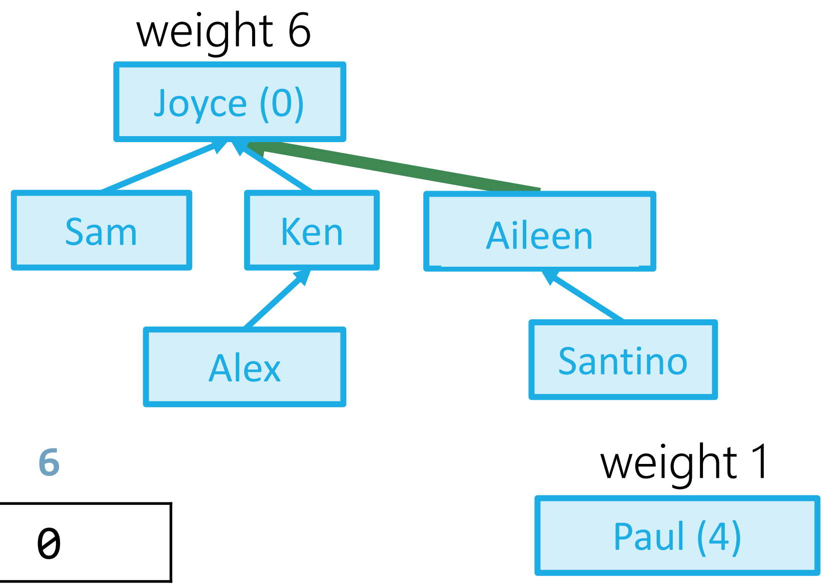
0	1	2	3	4	5	6
-4	0	-2	6	-1	2	0
Joyce	*Sam*	*Aileen*	*Alex*	*Paul*	*Santino*	*Ken*

Using Arrays: Union

For WeightedQuickUnion, we need to store the number of nodes in each tree (the weight)

Instead of just storing -1 to indicate a root, we can store -1 * weight!

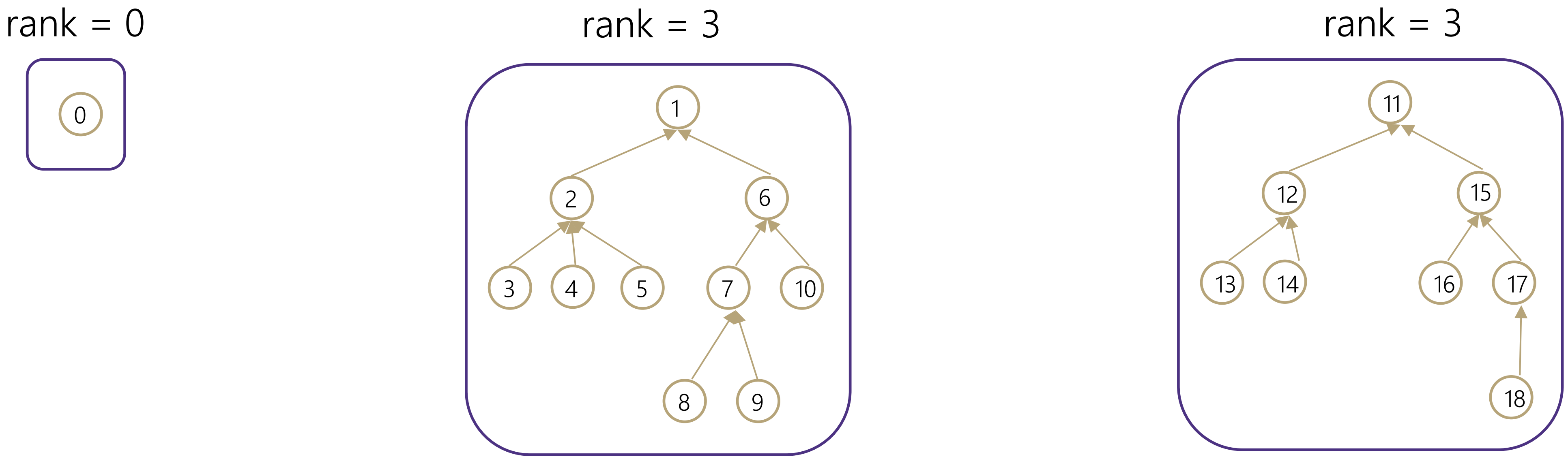
```
union(A, B):
  rootA = find(A)
  rootB = find(B)
  use -1 * array[rootA] and -1 *
       array[rootB] to determine weights
  put lighter root under heavier root
```



`union(Ken, Santino)`

0	1	2	3	4	5	6
-6	0	0	6	-1	2	0
Joyce	*Sam*	*Aileen*	*Alex*	*Paul*	*Santino*	*Ken*

Array Implementation

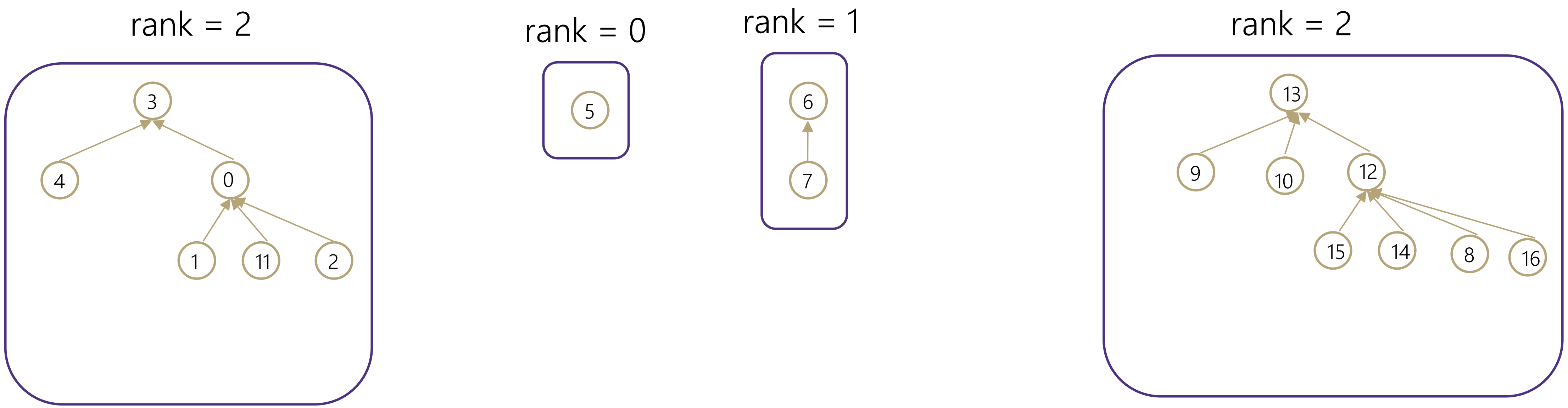


0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
-1	-4	1	2	2	2	1	6	7	7	6	-4	11	12	12	11	15	15	17

Store (rank * -1) - 1

Each "node" now only takes 4 bytes of memory instead of 32

Practice



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3	0	0	-3	3	-1	-2	6	12	13	13	0	13	-3	12	12	12

Using Arrays for WQU+PC

Same asymptotic runtime as using tree nodes, but check out all these other benefits:

- More compact in memory
- Better spatial locality, leading to better constant factors from cache usage
- Simplify the implementation!

	(Baseline)	QuickFind	QuickUnion	WeightedQuickUnion	WQU + Path Compression	ArrayWQU+PC
`makeSet(value)`	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
`find(value)`	$\Theta(n)$	$\Theta(1)$	$\Theta(n)$	$\Theta(\log n)$	$O(\log^* n)$	$O(\log^* n)$
`union(x, y)` `assuming root args`	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
`union(x, y)`	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(\log n)$	$O(\log^* n)$	$O(\log^* n)$

Appendix